Visual Justification of the Reciprocal Rule - Sharing

There are several ways to assign meaning to the division of numbers. One of them is the idea of sharing. We can, for example, distribute 15 apples into 3 bags. The result of the calculation $15 \div 3$ then indicates how many apples are in each bag.

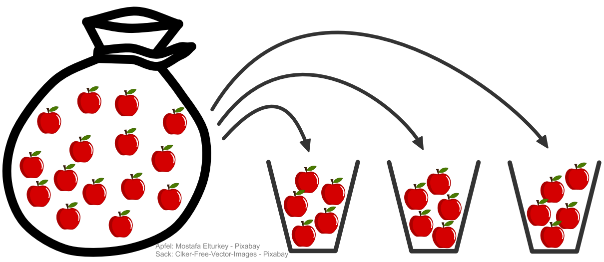

Abb. 1 Distributing 15 apples into 3 bags

If we replace the apples with water, we obtain a very straightforward way to understand the division of fractions: If we have a container of water and pour the water into a wider container - whereby we then distribute the water over a larger base area - the water level is lower. If we pour the water into a narrower container - whereby we then distribute the water over a smaller base area - the water level rises.

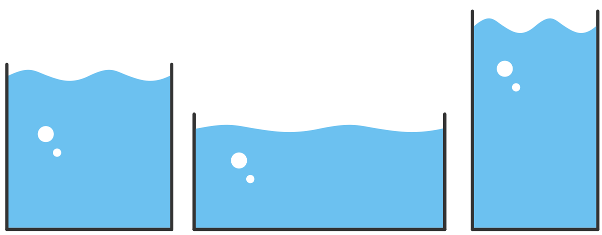

Abb. 2 Water level and container width

Instead of water, we now want to distribute areas. To do this, we color three quarters of the area of a unit square that is, a square with side length 1 green and distribute it over two wholes.

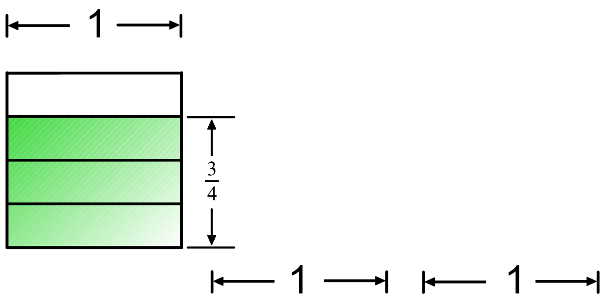

Abb. 3 $\frac{3}{4}$ over 2

Since we want to distribute over *two* wholes, we divide each quarter into *two* equal parts. The height of each of these parts is then $\frac{1}{8}$, because $\frac{1}{8}$ is half of $\frac{1}{4}$.

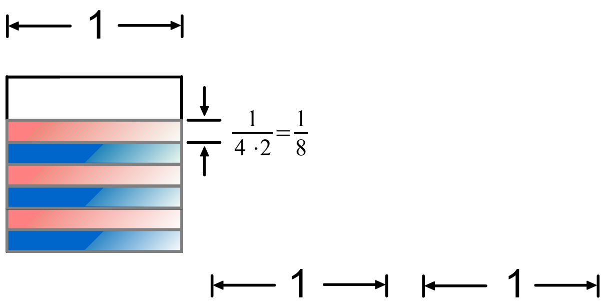

Abb. 4 Half of $\frac{1}{4}$ is $\frac{1}{8}$.

Now we can distribute the first quarter over two wholes.

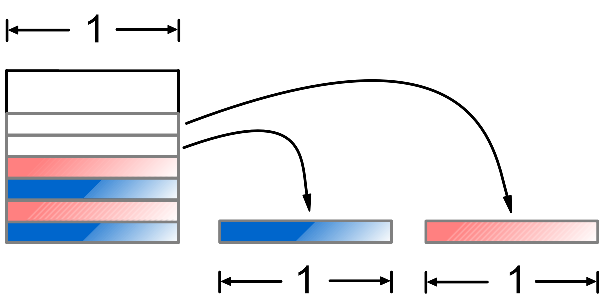

Abb. 5 Distributing $\frac{1}{4}$ over 2 wholes

If we also distribute the remaining quarters, two columns are formed, each with a height of $3 \times \frac{1}{8}$.

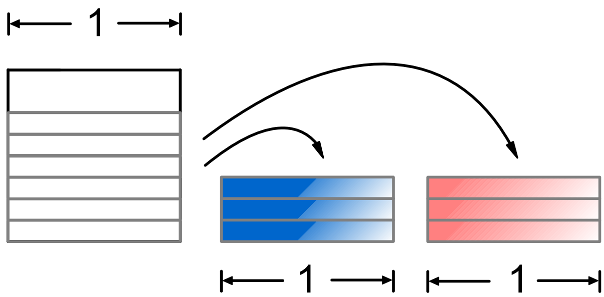

Abb. 6 Distributing $\frac{3}{4}$ over 2 wholes

As a calculation, we can express this as follows:

$$\frac{3}{4} \div 2 = \frac{3}{4} \div \frac{2}{1} = \frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{4 \times 2} = \frac{3}{8}$$

Now we want to work with properfractions and distribute four fifths over two thirds, that is, determine the result of $\frac{4}{5} \div \frac{2}{3}$.

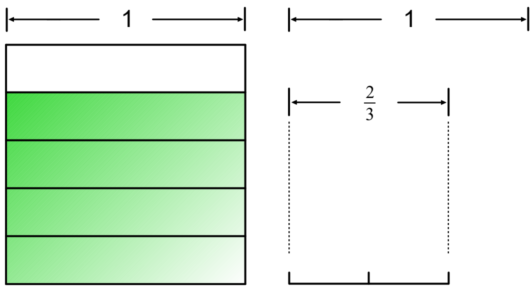

Abb. 7 Distributing $\frac{4}{5}$ over $\frac{2}{3}$

To do this, we divide the fifths into *three* parts each, so that what we want to distribute fits well over the *thirds*.

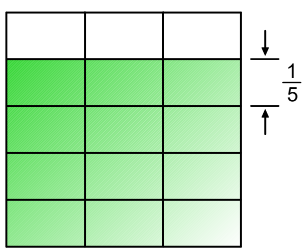

Abb. 8 Dividing $\frac{4}{5}$ into thirds

Since we want to distribute over *two* thirds, we divide each small rectangle into *two* equal parts. Each of these parts then has a height of $\frac{1}{5 \times 2} = \frac{1}{10}$.

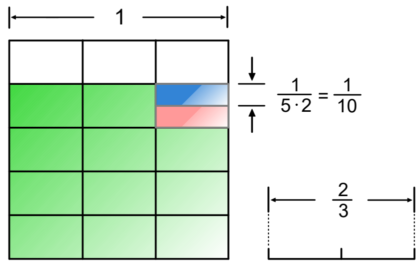

Abb. 9 Dividing one small unit into two parts

Now we can distribute.

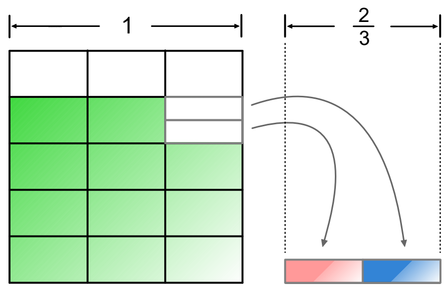

Abb. 10 Distributing one small unit over two thirds

We can repeat this process $4 \cdot 3 = 12$ times.

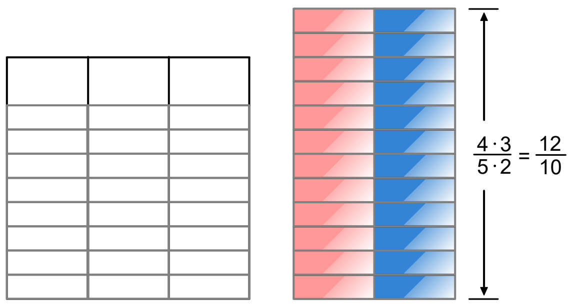

Abb. 11 Distributing $\frac{4}{5}$ over $\frac{2}{3}$

This column has a height of twelve tenths. They are *tenths*, because we divided the given **fifths** each into **two** parts, since we wanted to distribute over **two** of the thirds. They are *twelve* tenths, because we divided all the given **four** fifths into **three** small rectangles each, since we wanted to distribute over **thirds**.

$$\frac{4}{5} \div \frac{2}{3} = \frac{4 \times 3}{5 \times 2}$$