1 Dividing Fractions

1.1 Definition

In mathematics, the following rule has become standard:

> To divide by a fraction,
> multiply by the reciprocal.

Using variables, we can write this very succinctly:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$$

Here, a, b, c, and d are natural numbers, with b and d greater than 0.

Examples

$\frac{1}{2} \div \frac{1}{3} = \frac{1 \times 3}{2 \times 1} = \frac{3}{2}$

Unlike division of natural numbers, the result can be greater than the number being divided.

$\frac{2}{3} \div \frac{4}{5} = \frac{2 \times 5}{3 \times 4} = \frac{10}{12} = \frac{5}{6}$

The fraction that is being divided is called the **dividend**. The fraction by which we divide is called the **divisor**.

$\frac{4}{1} \div \frac{6}{1} = \frac{4 \times 1}{1 \times 6} = \frac{4}{6} = \frac{2}{3}$

If we apply fraction division to natural numbers, we obtain the results we are familiar with from arithmetic with natural numbers, because $\frac{4}{1}$ equals 4, $\frac{6}{1}$ equals 6, and the result of $4 \div 6$ is $\frac{2}{3}$.

1.2 Explaining the Division by Fractions

What we humans want to understand by the division by fractions is something we must decide for ourselves. There is nothing in nature that forces us to assign any particular meaning to the division of fractions. We now look at one way to make sense of the definition given above.

One way to consider what division by fractions might mean is to extend the properties of division of natural numbers into the realm of fractions. This approach will be pursued further below.

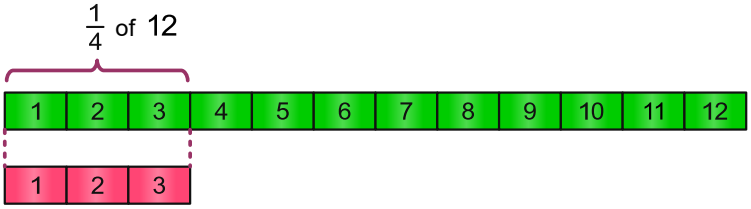
When we divide 12 by 3, we get 4 because 3 fits into 12 four times.
When we divide 3 by 12, we do not ask how many times 12 fits into 3, instead, we ask:

- Which part of 12 fits into 3?

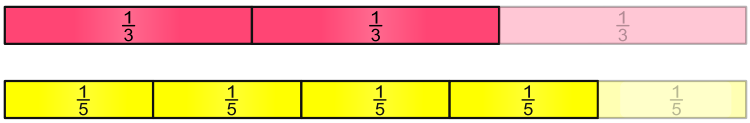
or alternatively:

- How much of 12 fits into 3?

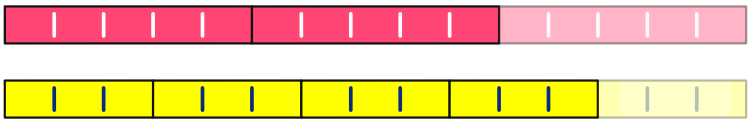
The answer is $\frac{1}{4}$, because one quarter of 12 equals 3.



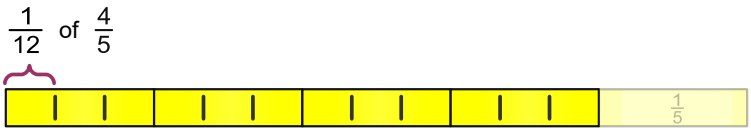
When we divide fractions, we ask the same question. For example, we can formulate the problem $\frac{2}{3} \div \frac{4}{5}$ as: Which part of $\frac{4}{5}$ fits into $\frac{2}{3}$?
To help visualize the calculation, we lay out fraction strips.



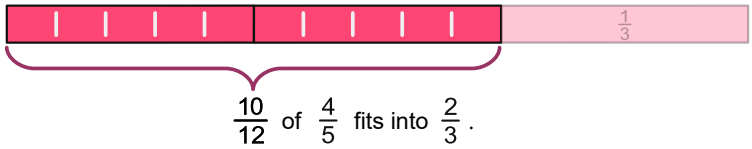
To answer the question of how much of $\frac{4}{5}$ fits into $\frac{2}{3}$, we divide the fractions into smaller units. We divide each third into 5 units, and each fifth into 3 units.
All of the small units now have the same size.



On the 4 fifths, there are $3 \times 4 = 12$ units.
Therefore, each unit is $\frac{1}{12}$ of $\frac{4}{5}$.
To obtain the denominator 12 of the result, we multiply the denominator of the dividend by the numerator of the divisor. This is one part of the reciprocal rule.



On the 2 thirds, there are $2 \times 5 = 10$ units. Therefore $\frac{10}{12}$ of $\frac{4}{5}$ fits into $\frac{2}{3}$.
To obtain the numerator 10 of the result, we multiply the numerator of the dividend by the denominator of the divisor. This is the other part of the reciprocal rule.



We write the calculation as follows:

$$\frac{2}{3} \div \frac{4}{5} = \frac{2 \times 5}{3 \times 4} = \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

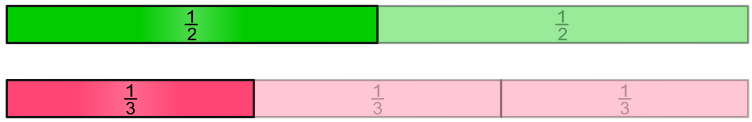
So, as a result, $\frac{5}{6}$ of $\frac{4}{5}$ fits into $\frac{2}{3}$.

Language note: In this text, *fit* is used in the singular form in this context, because, for example, $\frac{4}{5}$ is understood as a single number, not as four concrete objects.

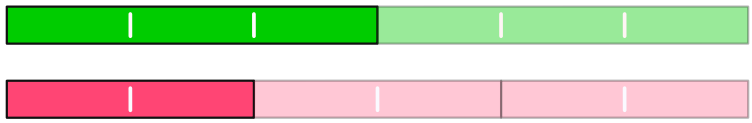
1.3 Examples

Example 1

What is $\frac{1}{2} \div \frac{1}{3}$? In other words: How much of $\frac{1}{3}$ fits into $\frac{1}{2}$?

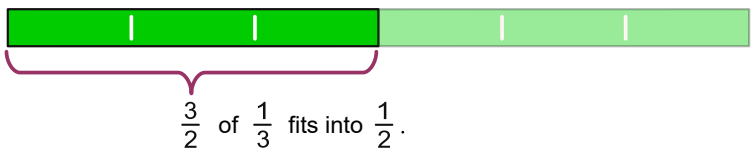


We divide one half into 3 units, and one third into 2 units.



We have divided one third into $2 \times 1 = 2$ units. So each unit is $\frac{1}{2}$ of $\frac{1}{3}$.

We have divided one half into $1 \times 3 = 3$ units. So $\frac{3}{2}$ of $\frac{1}{3}$ fits into $\frac{1}{2}$.

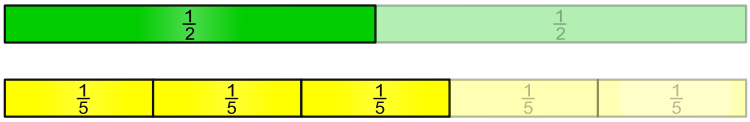


The calculation:

$$\frac{1}{2} \div \frac{1}{3} = \frac{1 \times 3}{2 \times 1} = \frac{3}{2}$$

Example 2

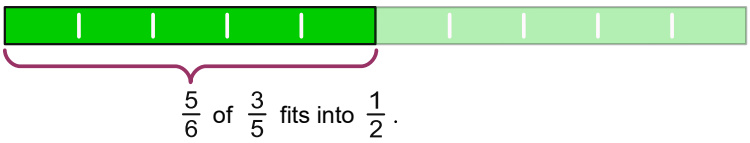
What is $\frac{1}{2} \div \frac{3}{5}$? In other words: How much of $\frac{3}{5}$ fits into $\frac{1}{2}$?



We divide one half into 5 units, and each fifth into 2 units.

We have divided the fifths into $2 \times 3 = 6$ units. Therefore, each unit is $\frac{1}{6}$ of $\frac{3}{5}$.

We have divided one half into $1 \times 5 = 5$ units. Therefore $\frac{5}{6}$ of $\frac{3}{5}$ fits into $\frac{1}{2}$.

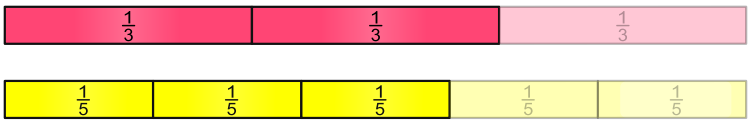


The calculation:

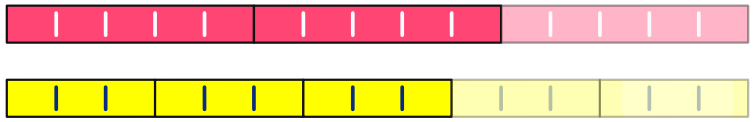
$$\frac{1}{2} \div \frac{3}{5} = \frac{1 \times 5}{2 \times 3} = \frac{5}{6}$$

Example 3

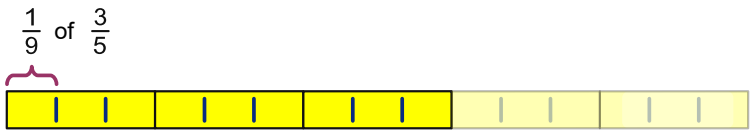
What is $\frac{2}{3} \div \frac{3}{5}$? In other words: How much of $\frac{3}{5}$ fits into $\frac{2}{3}$?



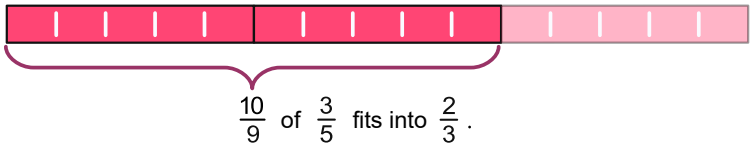
We divide each third into 5 units, and each fifth into 3 units.



We have divided the fifths into $3 \times 3 = 9$ units. Therefore each unit is $\frac{1}{9}$ of $\frac{3}{5}$.



We have divided the thirds into $2 \times 5 = 10$ units. Therefore, $\frac{10}{9}$ of $\frac{3}{5}$ fits into $\frac{2}{3}$.

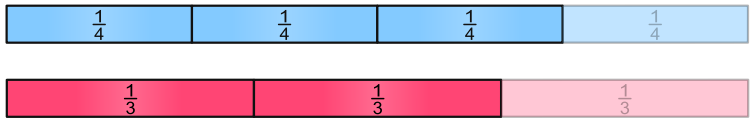


The calculation:

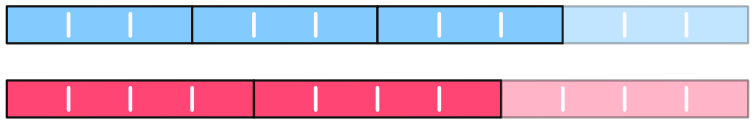
$$\frac{2}{3} \div \frac{3}{5} = \frac{2 \times 5}{3 \times 3} = \frac{10}{9}$$

Example 4

What is $\frac{3}{4} \div \frac{2}{3}$? In other words: How much of $\frac{2}{3}$ fits into $\frac{3}{4}$?

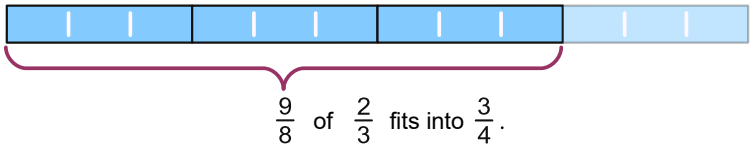


We divide each fourth into 3 units, and each third into 4 units.



We have divided the thirds into $4 \times 2 = 8$ units. Therefore, each unit is $\frac{1}{8}$ of $\frac{2}{3}$.

We have divided the fourths into $3 \times 3 = 9$ units. Therefore, $\frac{9}{8}$ of $\frac{2}{3}$ fits into $\frac{3}{4}$.

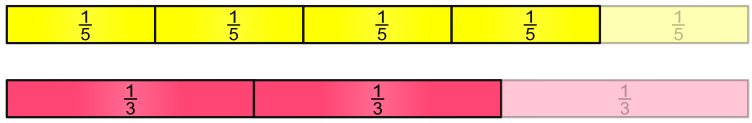


The calculation:

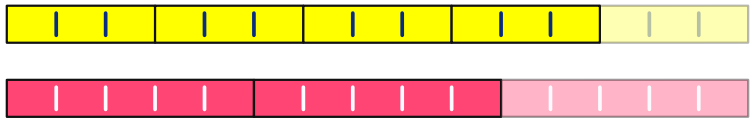
$$\frac{3}{4} \div \frac{2}{3} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8}$$

Example 5

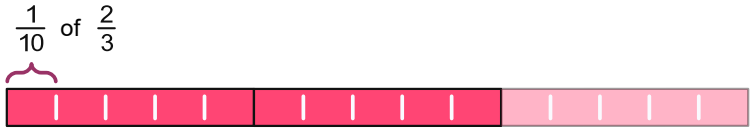
What is $\frac{4}{5} \div \frac{2}{3}$? In other words: How much of $\frac{2}{3}$ fits into $\frac{4}{5}$?



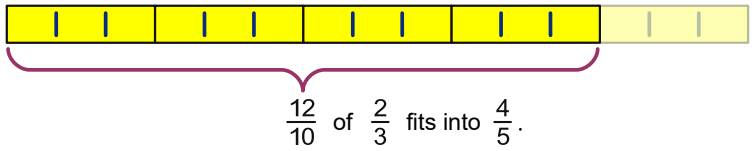
We divide each fifth into 3 units, and each third into 5 units.



We have divided the thirds into $5 \times 2 = 10$ units. Therefore, each unit is $\frac{1}{10}$ of $\frac{2}{3}$.



We have divided the fifths into $4 \times 3 = 12$ units. Therefore, $\frac{12}{10}$ of $\frac{2}{3}$ fits into $\frac{4}{5}$.

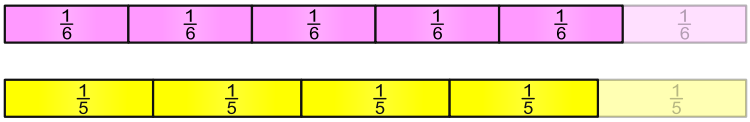


The calculation:

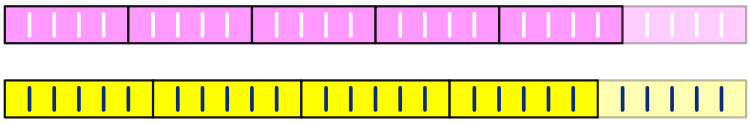
$$\frac{4}{5} \div \frac{2}{3} = \frac{4 \times 3}{5 \times 2} = \frac{12}{10} = \frac{12 \div 2}{10 \div 2} = \frac{6}{5}$$

Example 6

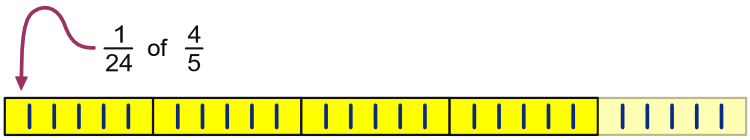
What is $\frac{5}{6} \div \frac{4}{5}$? In other words: How much of $\frac{4}{5}$ fits into $\frac{5}{6}$?



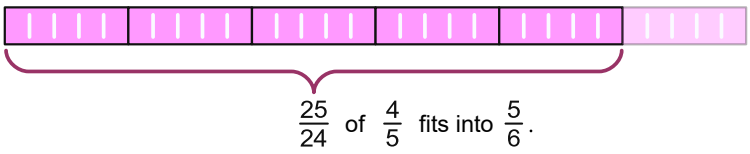
We divide each sixth into 5 units, and each fifth into 6 units.



We have divided the fifths into $6 \times 4 = 24$ units. Therefore, each unit is $\frac{1}{24}$ of $\frac{4}{5}$.



We have divided the sixths into $5 \times 5 = 25$ units. Therefore, $\frac{25}{24}$ of $\frac{4}{5}$ fits into $\frac{5}{6}$.

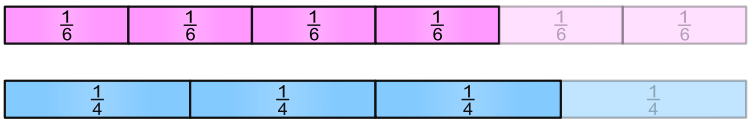


The calculation:

$$\frac{5}{6} \div \frac{4}{5} = \frac{5 \times 5}{6 \times 4} = \frac{25}{24}$$

Example 7

We want to divide $\frac{4}{6}$ by $\frac{3}{4}$.

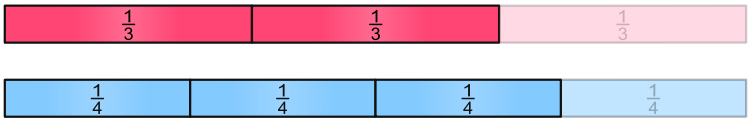


First, we simplify the fractions if possible. Thus:

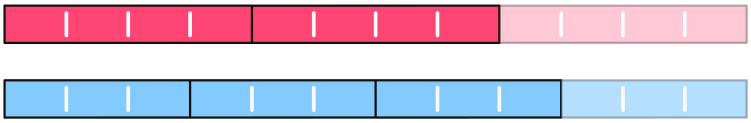
$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

The problem now is:

What is $\frac{2}{3} \div \frac{3}{4}$? In other words: How much of $\frac{3}{4}$ fits into $\frac{2}{3}$?

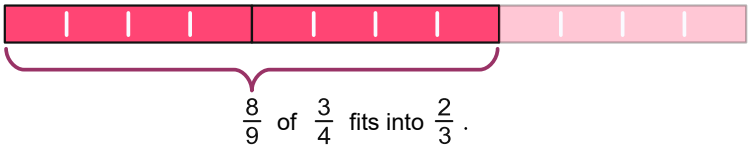


We divide each third into 4 units, and each fourth into 3 units.



We have divided the fourths into $3 \times 3 = 9$ units. Therefore, each unit is $\frac{1}{9}$ of $\frac{3}{4}$.

We have divided the thirds into $2 \times 4 = 8$ units. Therefore, $\frac{8}{9}$ of $\frac{3}{4}$ fits into $\frac{2}{3}$.

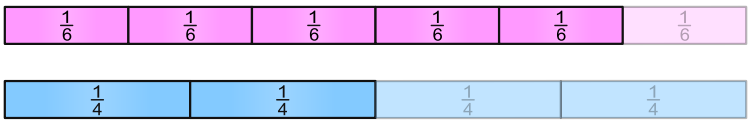


The calculation:

$$\frac{4}{6} \div \frac{3}{4} = \frac{4 \div 2}{6 \div 2} \div \frac{3}{4} = \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}$$

Example 8

We want to divide $\frac{5}{6}$ by $\frac{2}{4}$.

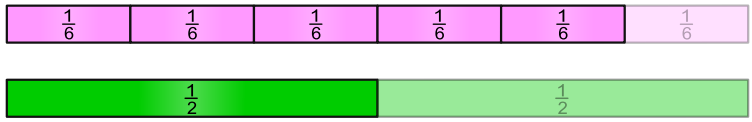


First, we simplify the fractions if possible. Thus:

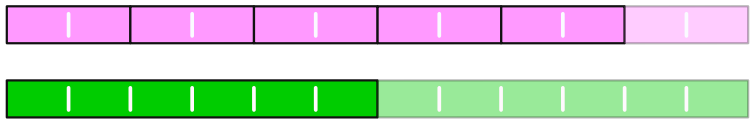
$$\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$$

The problem now is:

What is $\frac{5}{6} \div \frac{1}{2}$? In other words: How much of $\frac{1}{2}$ fits into $\frac{5}{6}$?

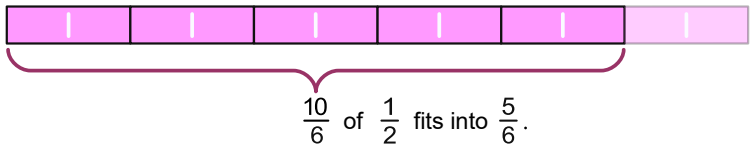


We divide each sixth into 2 units, and we divide one half into 6 units.



We have divided one half into $6 \times 1 = 6$ units. Therefore, each unit is $\frac{1}{6}$ of $\frac{1}{2}$.

We have divided the thirds into $5 \times 2 = 10$ units. Therefore, $\frac{10}{6}$ of $\frac{1}{2}$ fits into $\frac{5}{6}$.



The calculation:

$$\frac{5}{6} \div \frac{1}{2} = \frac{5 \times 2}{6 \times 1} = \frac{10}{6} = \frac{10 \div 2}{6 \div 2} = \frac{5}{3}$$