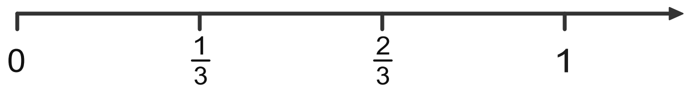
Fractions

What Fractions Look Like and What They Can Do

When we divide a whole into several equal parts, we get numbers that we call fractions. Our basic model is the segment from 0 to 1 on the number line, which we can divide, for example, into three equal parts.

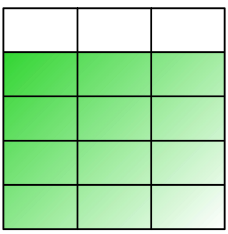
This gives us the fractions $\frac{1}{3}$ and $\frac{2}{3}$.



Fractions can also appear in many other contexts. Here are a few examples:

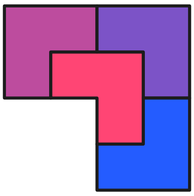
1. Fractions as Parts of a Whole

If we divide a round pizza into 8 equal slices, each slice represents $\frac{1}{8}$ of the whole pizza. The same principle works with other circular objects and geometric figures such as circles.



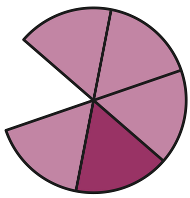
If we divide a rectangle into 15 equal smaller rectangles, each of the small rectangles represents $\frac{1}{15}$ of the large one. Here, $\frac{12}{15}$ are shaded green.

We only describe parts of a whole using fractions when all the parts are equal in size. The orange areas shown on the left are not considered examples of fractions.

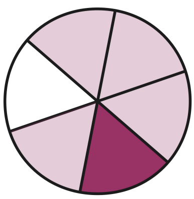


The division into equal parts can look “strange” — that’s perfectly fine. As long as all parts are the same size, we treat them as valid examples of fractions. Each colored area represents $\frac{1}{4}$ of the total area.

Sometimes the situation is ambiguous. How can we interpret the figure on the left?



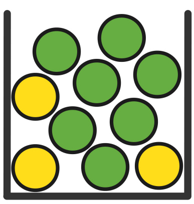
Is the whole the burgundy-colored area, of which $\frac{1}{5}$ is shaded dark?



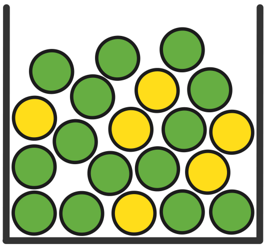
Or is the whole the entire circle, with $\frac{1}{6}$ missing?

2. Fractions as Parts of a Set

A fraction can describe how large one part of a whole is in comparison to the whole.

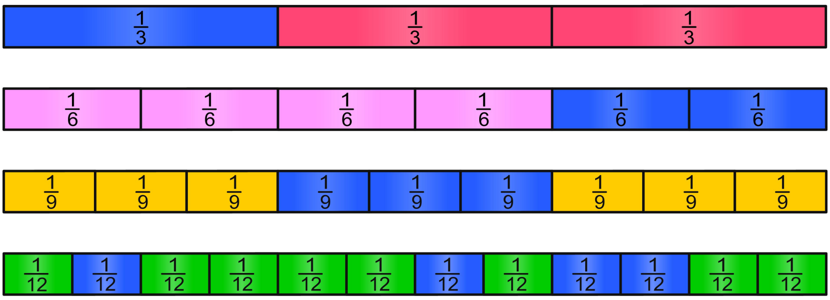


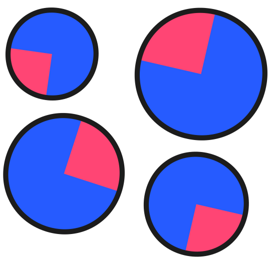
The fraction of green balls in the set is $\frac{7}{10}$ because 7 out of the 10 balls are green. The fraction of yellow balls is $\frac{3}{10}$ because 3 out of the 10 balls are yellow.



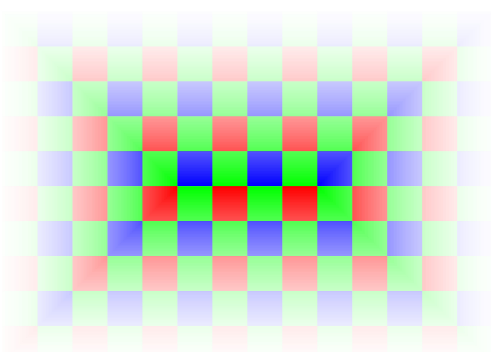
Now there are more balls in the container, but the fractions of green and yellow balls have remained the same. The fraction of green balls is still $\frac{7}{10}$ because 14 out of 20 balls are green. The fraction of yellow balls is $\frac{3}{10}$ because 6 out of 20 balls are yellow.

The fraction of the area shaded in blue in each fraction strip is the same in every strip. In the top strip, the blue area represents one third because one out of the three thirds is shaded blue. In the second strip, the blue area is also one third because two out of six sixths are blue, and so on.





The circular areas shown on the left are different in size. However, the fraction of the red area compared to the total area of each circle is the same. In each case, it is $\frac{1}{4}$.



Even if we do not know the exact size of an area, we can sometimes recognize from its structure what fraction is colored red, yellow, or blue. The fraction of the area that is red or blue is each $\frac{1}{4}$, and the green area makes up $\frac{1}{2}$. (This image shows the Bayer pattern, which corresponds to the arrangement of color filters in image sensors of digital cameras.)

Especially when parts cannot be directly seen, it can be useful to describe them using fractions.

Food is not only judged by taste and appearance — we can also describe what it contains. It is common to distinguish between protein, fat, and carbohydrates as nutritional components.

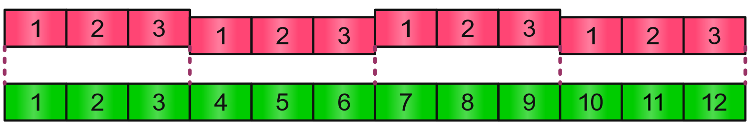
A pepperoni pizza typically contains about $\frac{11}{100}$ protein (read: "eleven hundredths"), about $\frac{10}{100}$ fat, and about $\frac{31}{100}$ carbohydrates (read: "thirty-one hundredths").

Nutritional information for tiramisu:

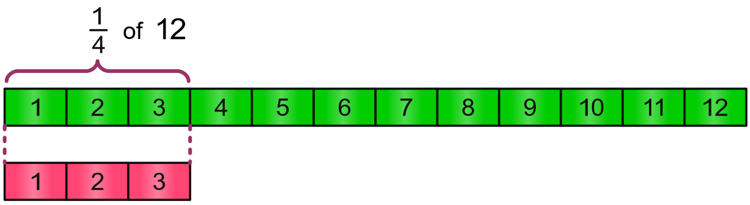
Protein	approx. $\frac{8}{100}$
Fat	approx. $\frac{13}{100}$
Carbohydrates	approx. $\frac{39}{100}$

3. Fractions as Results of Division

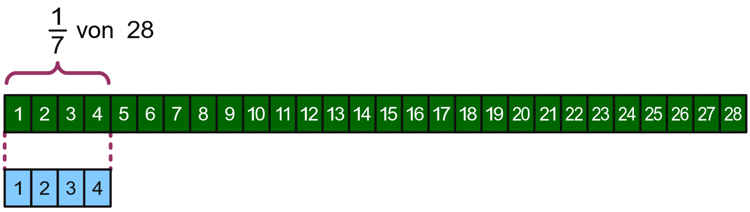
When we divide 12 by 3, we get 4. This result can be understood as the answer to the question: How many times does 3 fit into 12? And the answer is 4, because 3 fits into 12 four times.



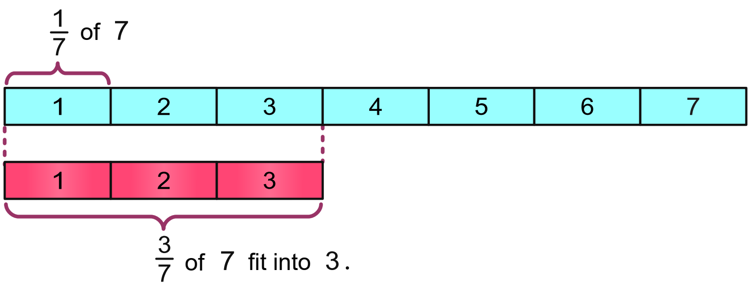
Using our new numbers, we can express this relationship in another way: Since 3 fits into 12 four times, 3 is one fourth of 12.
With fractions, we can now not only divide 12 by 3, but also divide 3 by 12.
Since 12 is greater than 3, we can no longer ask how many times 12 fits into 3. But we can ask: what part of 12 fits into 3? The answer: $\frac{1}{4}$.

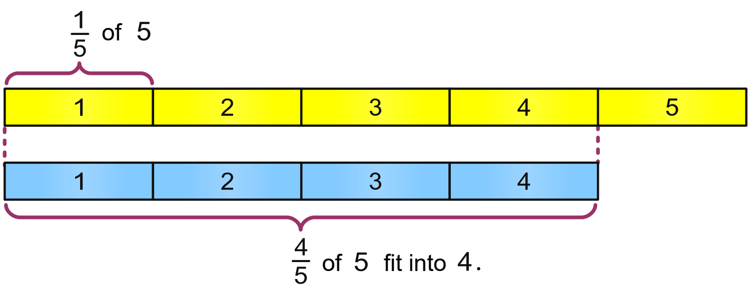


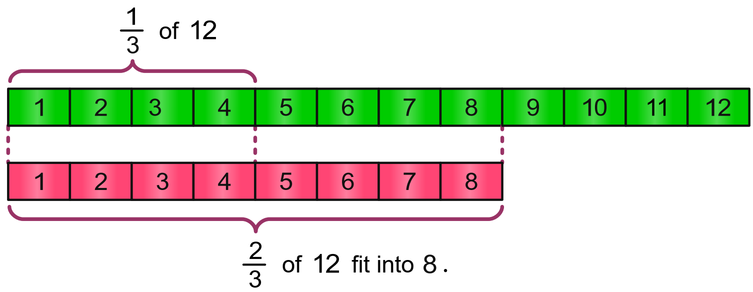
This works with other numbers as well. As we know, $7 \cdot 4 = 28$. So 28 is seven times as much as 4. Therefore, one seventh of 28 is equal to 4.



Up to now, when dividing a "smaller" number by a "larger" number, we've only used small numbers that were exact divisors of the larger number. But now we can divide any natural numbers — not just those that divide evenly. Let's look at some examples.

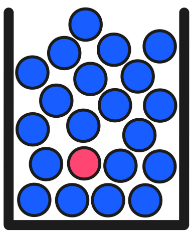






4. What Else Fractions Can Do

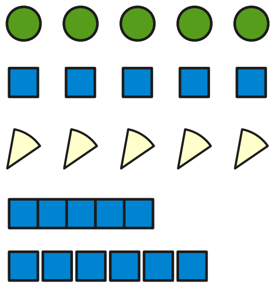
Fractions Describe Ratios



The ratio of red balls to blue balls in the box is 1 to 20. The fraction $\frac{1}{20}$ describes this ratio. If you draw one ball at random from the box, people often say the chance of drawing a red ball is "1 in 20." That's a common way to speak, but actually, $\frac{1}{20}$ is not the probability of drawing a red ball. In mathematics, we calculate probability by dividing the number of red balls — which is 1 — by the total number of balls — which is 21. So the actual probability of drawing a red ball is $\frac{1}{21}$.

To make concrete, cement is mixed with sand. The usual mixing ratio is given as 1 : 4 (read: "one to four"). The fraction $\frac{1}{4}$ describes this ratio. So if you're mixing concrete in a cement mixer, you might add one scoop of cement and then 4 scoops of sand. If that's not enough, you can repeat the process — one scoop of cement and 4 scoops of sand. However, the fraction of cement in the total mixture is not $\frac{1}{4}$, but rather $\frac{1}{5}$, because each batch consists of 5 scoops in total, with one scoop being cement.

Counting with Fractions

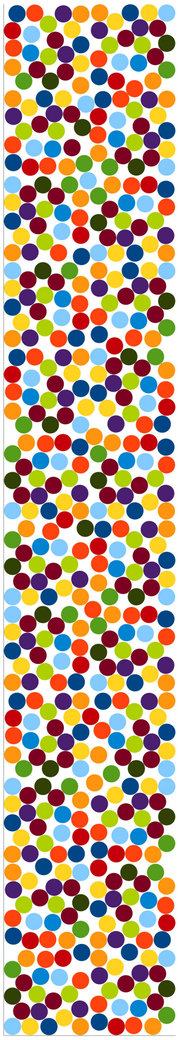


Counting with Fractions

With whole numbers, we can count many things. And we can also count things with fractions, if those things are parts of a whole. In the top row, we see 5 green circles. In the middle row, there are 5 blue squares, and in the bottom row there are 5 eighths. We can also write this as: $\frac{5}{8}$.
But we can also push the blue squares together to form one blue bar. Then, in the bottom row, we see $\frac{6}{5}$ of the blue bar.

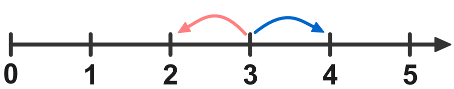
Using Fractions to Compare Sizes

Fractions allow us to compare sizes.
The child's height is $\frac{7}{11}$ of the woman's height. The difference in height is $\frac{4}{11}$ of the woman's height.

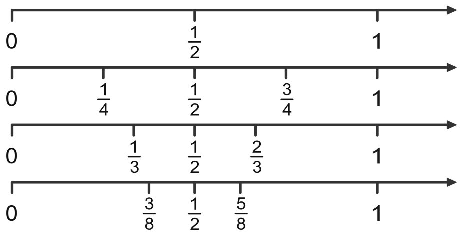


Can we also see fractions here?

Fractions without Immediate Successors or Predecessors



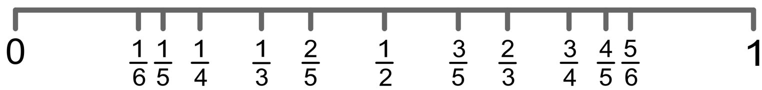
Every whole number has a successor on the number line. The successor of, for example, 3 is 4.
Every whole number greater than 0 also has a predecessor on the number line. The predecessor of, for example, 3 is 2.

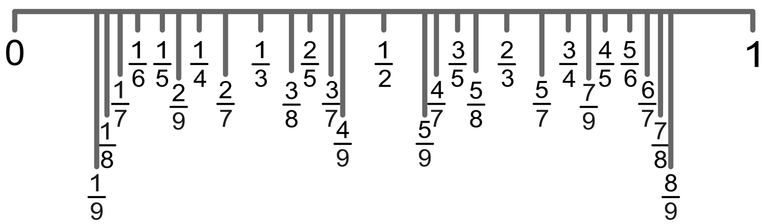


Fractions do not have an immediate successor or predecessor. For example, 1 is not the immediate successor of $\frac{1}{2}$, because $\frac{3}{4}$ lies between them. Likewise, $\frac{3}{4}$ is not the immediate successor of $\frac{1}{2}$, because $\frac{2}{3}$ lies between these two numbers—and so on.

Fractions are Dense

We can fit arbitrarily many fractions between the numbers 0 and 1. This is also true for any other segment of the number line. That is why we say fractions are dense on the number line.





Infinitely Many Fractions at a Single Point

Are there truly infinitely many fractions at points on the number line, or are there just arbitrarily many?