Fractions

A complete visual explanation of all fraction calculation rules based on fraction strips

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Contents

1 Fractions **3**
1.1 Definitions . . . 3
1.2 Properties of Fractions . . . 4
1.3 Fraction Strips . . . 7

2 Fractions - Meanings and Models **8**
2.1 What Fractions Look Like and What They Can Do . . . 8
2.2 Fractions as Parts of a Whole . . . 8
2.3 Fractions as Parts of a Set . . . 9
2.4 Fractions as Results of Division . . . 11
2.5 What Else Fractions Can Do . . . 13
2.5.1 Fractions Describe Ratios . . . 13
2.5.2 Counting with Fractions . . . 14
2.5.3 Using Fractions to Compare Sizes . . . 14
2.6 Fractions without Immediate Successors or Predecessors . . . 15
2.7 Fractions are Dense . . . 16
2.8 Infinitely Many Fractions at a Single Point . . . 16

3 Equivalent Fractions **17**
3.1 Creating Equivalent Fractions . . . 17
3.2 Visualizing Equivalent Fractions . . . 19

4 Simplifying Fractions **21**
4.1 Definition . . . 21
4.2 Examples . . . 22
4.3 Simplifying by all divisors of the denominator . . . 23
4.4 Simplifying as the inverse of expanding . . . 25
4.5 Simplifying a Fraction . . . 26
4.6 Examples . . . 27

5 Finding the Least Common Denominator **29**
5.1 Making Fractions Have the Same Denominator . . . 29
5.2 Simplify first . . . 31
5.3 Rewrite with the LCM . . . 32
5.4 Flowchart . . . 34
5.5 Examples . . . 36

6 Comparing Fractions **44**
6.1 Equivalent Fractions . . . 44
6.2 Comparing Fractions . . . 44

7 Adding Fractions **50**
7.1 Adding Like Fractions 50
7.2 Flowchart 54
7.3 Examples 55
7.4 Exercises 58

8 Subtracting Fractions **82**
8.1 Subtracting Like Fractions 82
8.2 Subtracting Unlike Fractions 83
8.3 Flowchart 84
8.4 Examples 85
8.5 Exercises 90

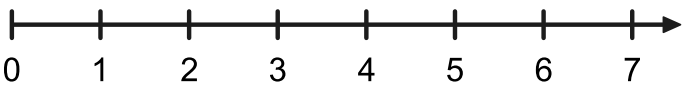
9 Multiplying Fractions **97**
9.1 Definition 97
9.2 Examples 97
9.3 Phrasing 98
9.4 Explaining Fraction Multiplication 98
9.5 Cross-Canceling 101
9.6 Swapping Numerators and Denominators 102
9.7 Examples 104

10 Dividing Fractions **108**
10.1 Definition 108
10.2 Explaining the Division by Fractions 108
10.3 Examples 110

1 Fractions

1.1 Definitions

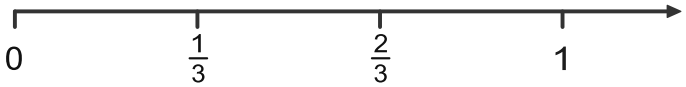
So far, we have worked with the numbers 0, 1, 2, 3, 4, 5, ... These numbers are called **natural numbers**, and we have represented them on the number line.



With natural numbers, we can count whole things like tomatoes, sunrises, or ideas. We can add, subtract, multiply, and divide these numbers.
A tomato can be divided into 4 equal parts. We can also perform arithmetic with these parts. If we combine two of these parts with two more, we get a whole tomato again. But it's not $2 + 2 = 1$.
So if we want to do arithmetic with parts of a whole, we need other numbers—numbers that lie on the number line between the natural numbers.

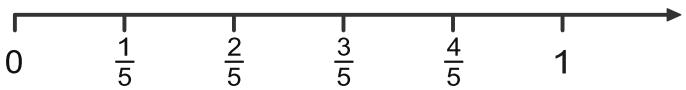
The number that lies exactly halfway between 0 and 1 is called: $\frac{1}{2}$ (read: one half).

If we divide a whole—that is, the segment between 0 and 1—into three equal parts, we get the numbers $\frac{1}{3}$ (read: one third) and $\frac{2}{3}$ (read: two thirds).

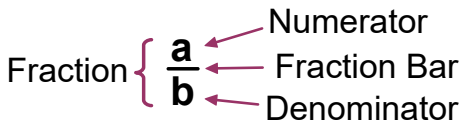


If we divide a whole into 4 equal parts, we get the numbers $\frac{1}{4}$ (read: one fourth), $\frac{2}{4}$ (read: two fourths), and $\frac{3}{4}$ (read: three fourths).

If we divide a whole into 5 equal parts, we get fifths.



In this way, we can create many more numbers. These numbers are called **fractions**. A **fraction** consists of a **fraction bar**, a whole number above the bar, and a whole number below the bar. The number above the bar is called the **numerator**, and the number below the bar is called the **denominator**.



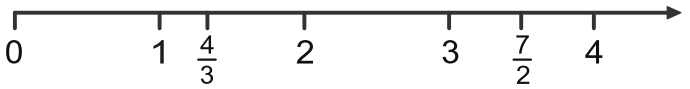
Definition: Fractions are numbers made up of equal parts of a whole. The denominator of a fraction tells us *what kind* of parts the fraction is made of, and the numerator tells us *how many* of those parts it consists of.

A fraction is also called a **rational number**.
All fractions together make up the **set of rational numbers**. We denote this set with the symbol $\mathbb{Q}$.

1.2 Properties of Fractions

1. Fractions Can Be Greater Than 1

Fractions can also be greater than 1. For example, the fractions $\frac{4}{3}$ and $\frac{7}{2}$ (read: seven halves) lie to the right of 1 on the number line.



2. Denominators Must Not Be Zero

We have defined fractions as numbers that arise when a whole is divided into equally sized parts. Since it is not possible to divide something into 0 parts, we stipulate that fractions with a denominator of 0 shall not exist. Even if we do not divide a whole at all, it still consists of *one* part — not of none at all.

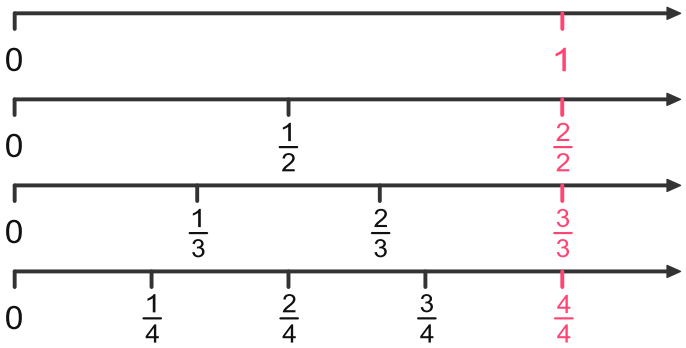
3. The Denominator Can Be Equal To 1

We define that the fraction $\frac{1}{1}$ (read: one whole) is equal to 1. Furthermore, $\frac{2}{1}$ (read: two wholes) is equal to 2, $\frac{3}{1}$ is equal to 3, and so on.

4. Multiple Numbers at the Same Place on the Number Line

If we divide a whole into

- 2 equal parts, then 2 of those parts are as big as the whole.
- 3 equal parts, then 3 of those parts are as big as the whole.
- 4 equal parts, then 4 of those parts are as big as the whole.

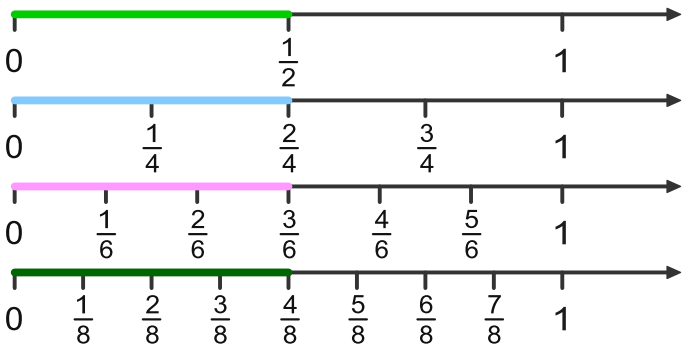


We therefore define: At the place on the number line where we find the 1, we also find other numbers. It holds that:

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = ...$$

There are also multiple numbers at other places on the number line.
If we divide a whole into

- 2 equal parts, then one of those parts is as big as $\frac{1}{2}$.
- 4 equal parts, then 2 of those parts are as big as $\frac{1}{2}$.
- 6 equal parts, then 3 of those parts are as big as $\frac{1}{2}$.
- 8 equal parts, then 4 of those parts are as big as $\frac{1}{2}$.



We see:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \ldots$$

The same applies to other fractions, such as:

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \ldots \text{ and}$$

$$\frac{3}{7} = \frac{6}{14} = \frac{9}{21} = \frac{12}{28} = \ldots .$$

We will explain these relationships when we talk about how to extend fractions.
But here's something we can already guess:
If the denominator of a fraction is twice the numerator,
then the fraction equals $\frac{1}{2}$.
If the denominator of a fraction is three times the numerator,
then the fraction equals $\frac{1}{3}$.

5. The Bigger the Denominator, the Smaller the Parts

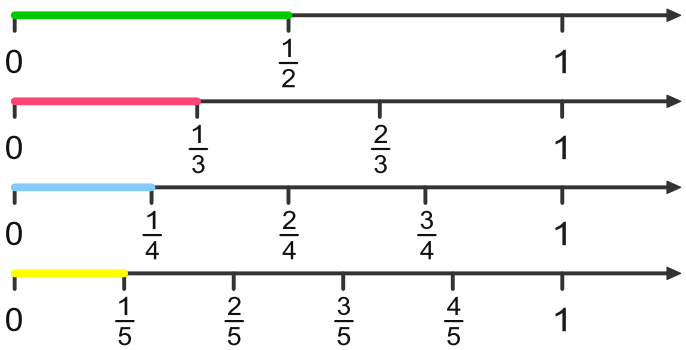
If we divide a whole into two equal parts, each of these parts is as big as $\frac{1}{2}$.
If we divide a whole into *more* than two equal parts, each of these parts is *smaller* than $\frac{1}{2}$. Therefore, for example:

$$\frac{1}{3} \text{ is smaller than } \frac{1}{2},$$

$$\frac{1}{4} \text{ is smaller than } \frac{1}{2},$$

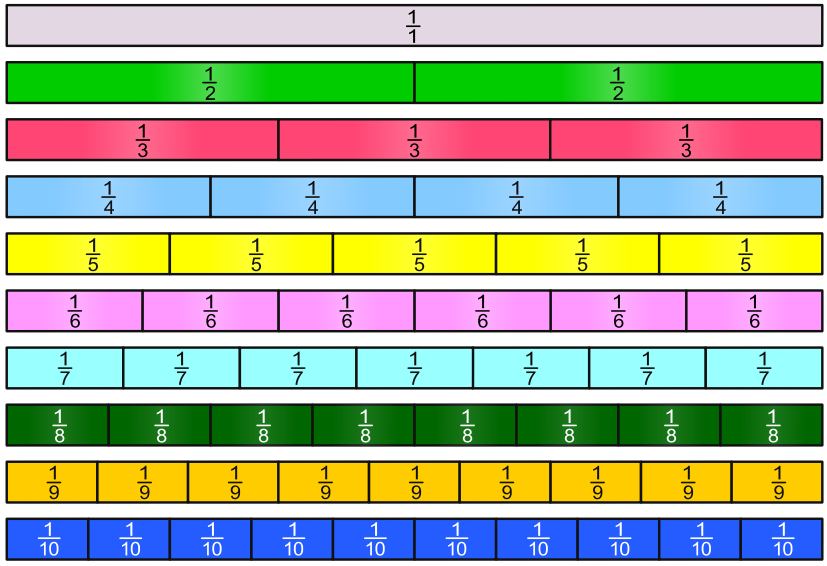
$$\frac{1}{5} \text{ is smaller than } \frac{1}{2}, \text{ and so on.}$$

If we divide a whole into more than three equal parts, each of these parts is *smaller* than $\frac{1}{3}$. For example, $\frac{1}{4}$, $\frac{1}{5}$, etc. are each smaller than $\frac{1}{3}$.

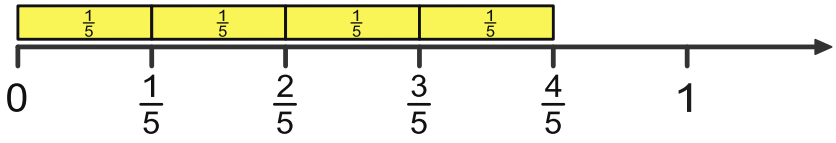


1.3 Fraction Strips

To help us visualize fractions and their properties, we use fraction strips. Here are the fraction strips up to tenths.



We can place the fraction strips on the number line.

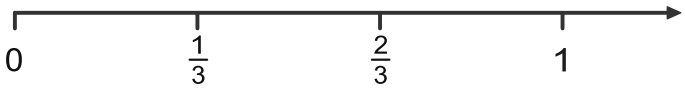


2 Fractions - Meanings and Models

2.1 What Fractions Look Like and What They Can Do

When we divide a whole into several equal parts, we get numbers that we call fractions. Our basic model is the segment from 0 to 1 on the number line, which we can divide, for example, into three equal parts.

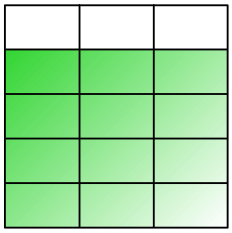
This gives us the fractions $\mathbf{\frac{1}{3}}$ and $\mathbf{\frac{2}{3}}$.



Fractions can also appear in many other contexts. Here are a few examples:

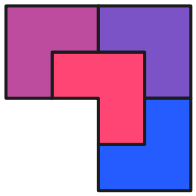
2.2 Fractions as Parts of a Whole

If we divide a round pizza into 8 equal slices, each slice represents $\frac{1}{8}$ of the whole pizza. The same principle works with other circular objects and geometric figures such as circles.



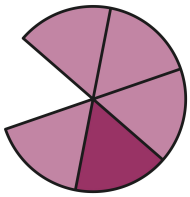
If we divide a rectangle into 15 equal smaller rectangles, each of the small rectangles represents $\frac{1}{15}$ of the large one. Here, $\frac{12}{15}$ are shaded green.

We only describe parts of a whole using fractions when all the parts are equal in size. The orange areas shown on the left are not considered examples of fractions.

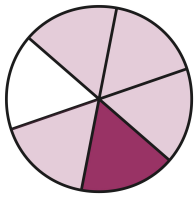


The division into equal parts can look "strange" — that's perfectly fine. As long as all parts are the same size, we treat them as valid examples of fractions. Each colored area represents $\frac{1}{4}$ of the total area.

Sometimes the situation is ambiguous. How can we interpret the figure on the left?



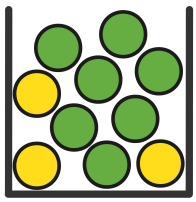
Is the whole the burgundy-colored area, of which $\frac{1}{5}$ is shaded dark?



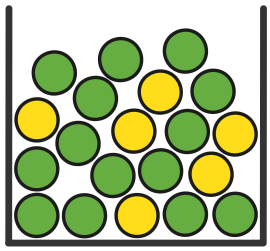
Or is the whole the entire circle, with $\frac{1}{6}$ missing?

2.3 Fractions as Parts of a Set

A fraction can describe how large one part of a whole is in comparison to the whole.

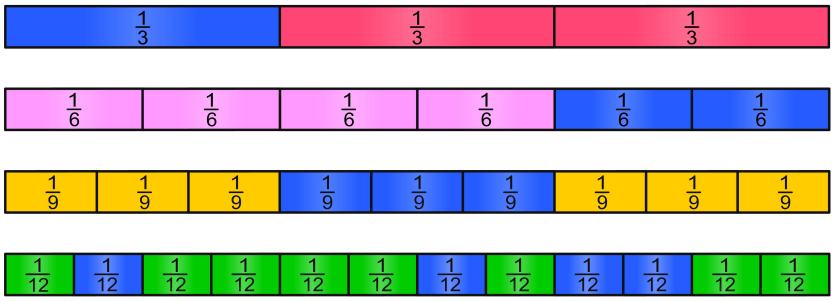


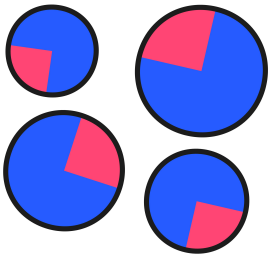
The fraction of green balls in the set is $\frac{7}{10}$ because 7 out of the 10 balls are green. The fraction of yellow balls is $\frac{3}{10}$ because 3 out of the 10 balls are yellow.



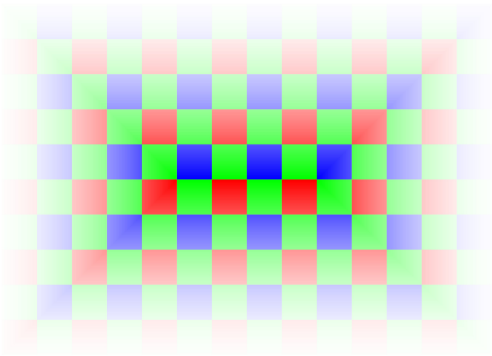
Now there are more balls in the container, but the fractions of green and yellow balls have remained the same. The fraction of green balls is still $\frac{7}{10}$ because 14 out of 20 balls are green. The fraction of yellow balls is $\frac{3}{10}$ because 6 out of 20 balls are yellow.

The fraction of the area shaded in blue in each fraction strip is the same in every strip. In the top strip, the blue area represents one third because one out of the three thirds is shaded blue. In the second strip, the blue area is also one third because two out of six sixths are blue, and so on.





The circular areas shown on the left are different in size. However, the fraction of the red area compared to the total area of each circle is the same.
In each case, it is $\frac{1}{4}$.



Even if we do not know the exact size of an area, we can sometimes recognize from its structure what fraction is colored red, yellow, or blue. The fraction of the area that is red or blue is each $\frac{1}{4}$, and the green area makes up $\frac{1}{2}$.
(This image shows the Bayer pattern, which corresponds to the arrangement of color filters in image sensors of digital cameras.)

Especially when parts cannot be directly seen, it can be useful to describe them using fractions.

Food is not only judged by taste and appearance — we can also describe what it contains. It is common to distinguish between protein, fat, and carbohydrates as nutritional components.

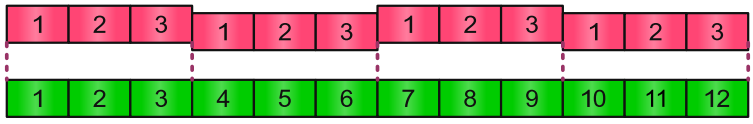
A pepperoni pizza typically contains about $\frac{11}{100}$ protein (read: "eleven hundredths"), about $\frac{10}{100}$ fat, and about $\frac{31}{100}$ carbohydrates (read: "thirty-one hundredths").

Nutritional information for tiramisu:

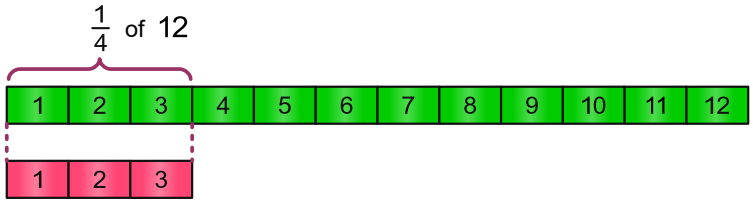
Protein	approx. $\frac{8}{100}$
Fat	approx. $\frac{13}{100}$
Carbohydrates	approx. $\frac{39}{100}$

2.4 Fractions as Results of Division

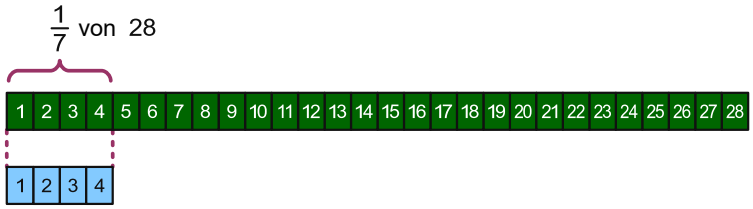
When we divide 12 by 3, we get 4. This result can be understood as the answer to the question: How many times does 3 fit into 12? And the answer is 4, because 3 fits into 12 four times.



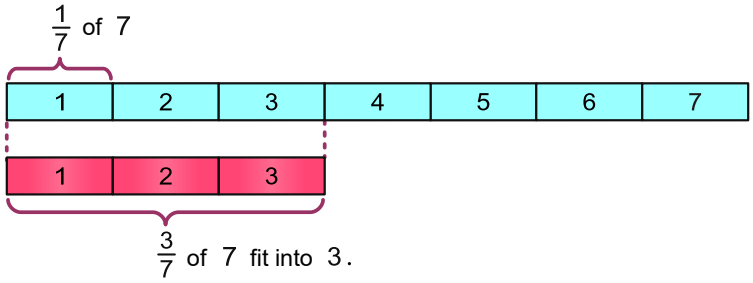
Using our new numbers, we can express this relationship in another way: Since 3 fits into 12 four times, 3 is one fourth of 12.
With fractions, we can now not only divide 12 by 3, but also divide 3 by 12.
Since 12 is greater than 3, we can no longer ask how many times 12 fits into 3.
But we can ask: what part of 12 fits into 3? The answer: $\frac{1}{4}$.

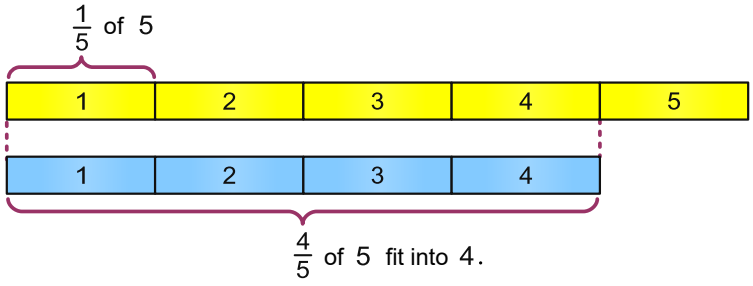


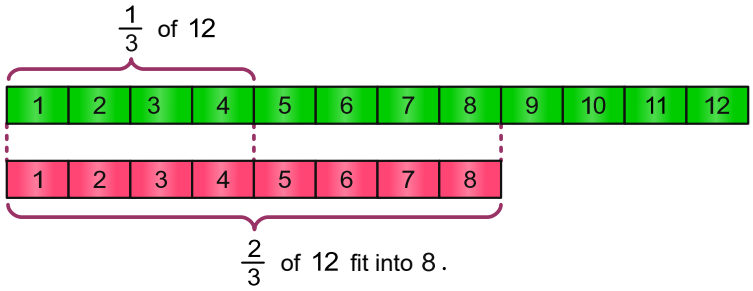
This works with other numbers as well. As we know, $7 \cdot 4 = 28$. So 28 is seven times as much as 4. Therefore, one seventh of 28 is equal to 4.



Up to now, when dividing a "smaller" number by a "larger" number, we've only used small numbers that were exact divisors of the larger number. But now we can divide any natural numbers — not just those that divide evenly. Let's look at some examples.

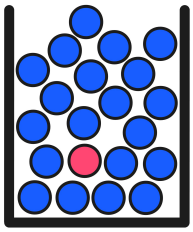






2.5 What Else Fractions Can Do

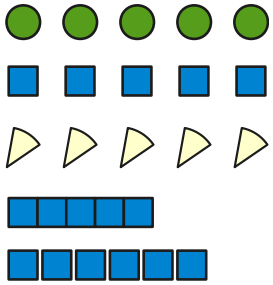
2.5.1 Fractions Describe Ratios



The ratio of red balls to blue balls in the box is 1 to 20. The fraction $\frac{1}{20}$ describes this ratio. If you draw one ball at random from the box, people often say the chance of drawing a red ball is "1 in 20."
That's a common way to speak, but actually, $\mathbf{\frac{1}{20}}$ is not the probability of drawing a red ball. In mathematics, we calculate probability by dividing the number of red balls — which is 1 — by the total number of balls — which is 21. So the actual probability of drawing a red ball is $\mathbf{\frac{1}{21}}$.

To make concrete, cement is mixed with sand. The usual mixing ratio is given as 1 : 4 (read: "one to four"). The fraction $\frac{1}{4}$ describes this ratio. So if you're mixing concrete in a cement mixer, you might add one scoop of cement and then 4 scoops of sand. If that's not enough, you can repeat the process — one scoop of cement and 4 scoops of sand. However, the fraction of cement in the total mixture is not $\frac{1}{4}$, but rather $\frac{1}{5}$, because each batch consists of 5 scoops in total, with one scoop being cement.

2.5.2 Counting with Fractions



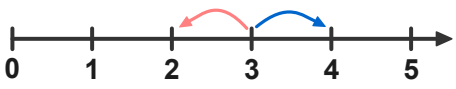
With whole numbers, we can count many things. And we can also count things with fractions, if those things are parts of a whole. In the top row, we see 5 green circles. In the middle row, there are 5 blue squares, and in the bottom row there are 5 eighths. We can also write this as: $\frac{5}{8}$.
But we can also push the blue squares together to form one blue bar. Then, in the bottom row, we see $\frac{6}{5}$ of the blue bar.

2.5.3 Using Fractions to Compare Sizes

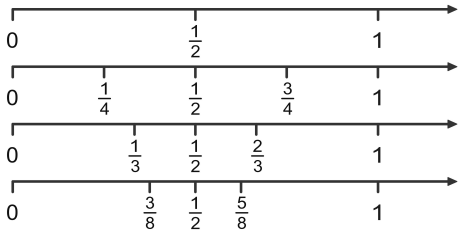
Fractions allow us to compare sizes.
The child's height is $\frac{7}{11}$ of the woman's height. The difference in height is $\frac{4}{11}$ of the woman's height.

Can we also see fractions here?

2.6 Fractions without Immediate Successors or Predecessors



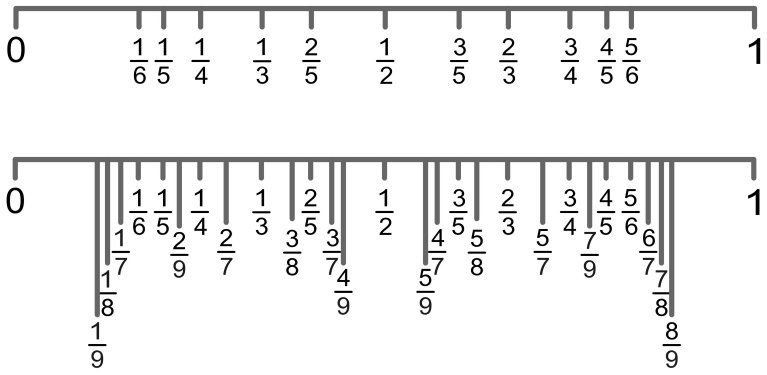
Every whole number has a successor on the number line. The successor of, for example, 3 is 4.
Every whole number greater than 0 also has a predecessor on the number line. The predecessor of, for example, 3 is 2.



Fractions do not have an immediate successor or predecessor. For example, 1 is not the immediate successor of $\frac{1}{2}$, because $\frac{3}{4}$ lies between them. Likewise, $\frac{3}{4}$ is not the immediate successor of $\frac{1}{2}$, because $\frac{2}{3}$ lies between these two numbers - and so on.

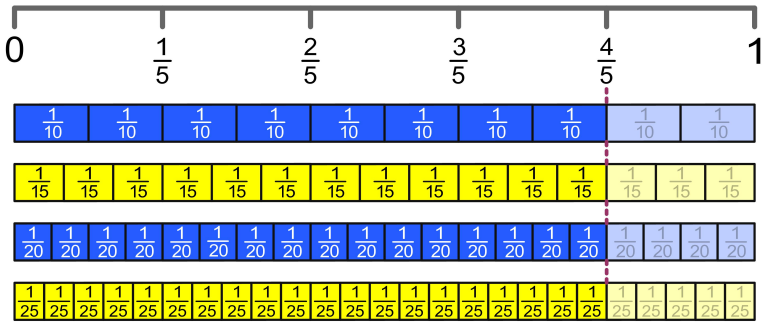
2.7 Fractions are Dense

We can fit arbitrarily many fractions between the numbers 0 and 1. This is also true for any other segment of the number line. That is why we say fractions are dense on the number line.



2.8 Infinitely Many Fractions at a Single Point

Are there truly infinitely many fractions at points on the number line, or are there just arbitrarily many?



3 Equivalent Fractions

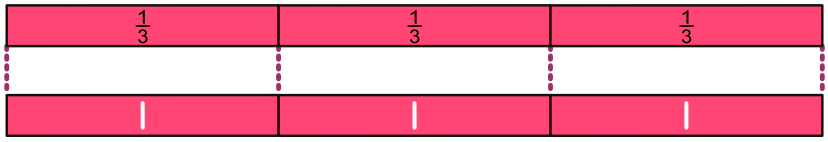
We can often make calculations with fractions easier by not working with the given fractions directly, but instead using other fractions that are equivalent in size.
That's why there are several procedures for finding a different fraction that is the same size as a given one. Here is one:

3.1 Creating Equivalent Fractions

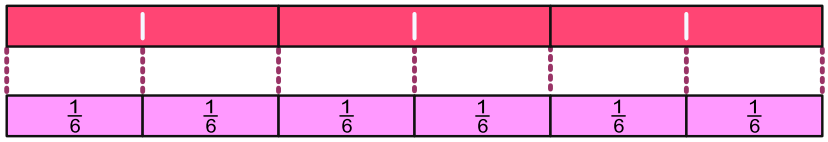
Arithmetically, we create an equivalent fraction by multiplying both the numerator and the denominator by the same natural number. Visually, we create an equivalent fraction by dividing the parts of the original fraction into smaller equal parts. Let's look at some examples.

Example 1

If we divide each third into two equal parts, there are $3 \times 2 = 6$ equal parts on this fraction strip.

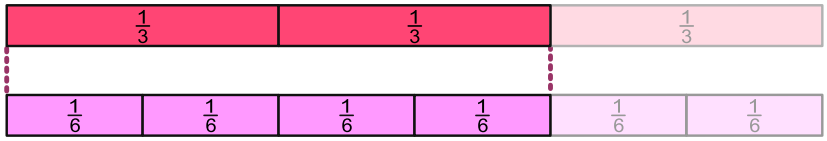


These are sixths.



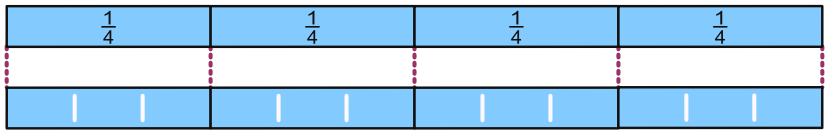
Since we divided each third into 2 equal parts, each of these parts is only half as large as $\frac{1}{3}$.
So for a fraction with denominator 6 that is supposed to be the same size as a fraction with denominator 3, we need twice as many parts.
Therefore, if we multiply the numerator and the denominator of $\frac{2}{3}$ by 2, we get an equivalent fraction, namely $\frac{4}{6}$.
We have:

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} \,,$$

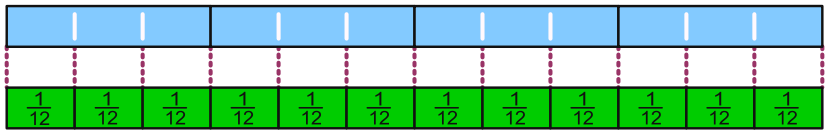


Example 2

If we divide each fourth into 3 equal parts, there are $4 \times 3 = 12$ equal parts on this fraction strip.

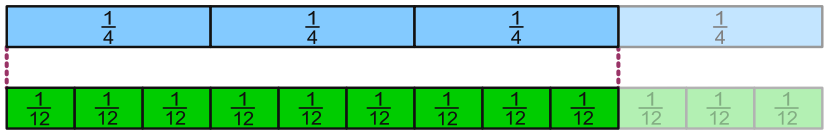


These are twelfths.



Since we divided each fourth into 3 equal parts, each of these parts is one third the size of $\mathbf{\frac{1}{4}}$.
So for a fraction with denominator 12 that is supposed to be the same size as a fraction with denominator 4, we need three times as many parts.
Therefore, if we multiply the numerator and the denominator of $\mathbf{\frac{3}{4}}$ by 3, we get an equivalent fraction, namely $\mathbf{\frac{9}{12}}$.
We have:

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12} \ .$$



What we saw above applies to all other fractions as well. The following statement holds for all natural numbers greater than 0.

> If both the numerator and the denominator of a fraction are multiplied by the same number, the result is a fraction of the same size.

We can also write this clearly using variables. If you want to make a general statement about a fraction in mathematics, you use variables - that is, letters that can stand for numbers. A general fraction is written, for example, as $\frac{\mathbf{a}}{\mathbf{b}}$. If it is to be multiplied by any natural number, that number is also written as a variable, for example **n**.
We can now write the statement above like this:
For any fraction $\frac{\mathbf{a}}{\mathbf{b}}$ and any natural number **n** (except 0), the following holds:

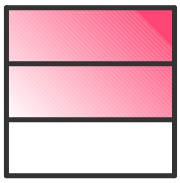
$$\frac{a}{b} = \frac{a \times n}{b \times n}$$

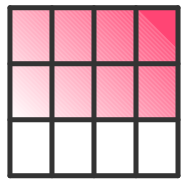
We call the process of multiplying both the numerator and the denominator of a fraction by the same number *creating an equivalent fraction.*

3.2 Visualizing Equivalent Fractions

There are many ways to visualize the idea of creating equivalent fractions.

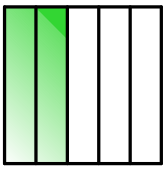
We can use a pizza to understand that the value of a fraction doesn't change when we divide each part into smaller equal parts: A pizza stays the same size, no matter whether it's divided into 2, 4, 6, 8, ... parts. So instead of eating one half of a pizza, we can also eat $\frac{\mathbf{2}}{\mathbf{4}}$, $\frac{\mathbf{3}}{\mathbf{6}}$, or $\frac{\mathbf{4}}{\mathbf{8}}$ of the pizza.

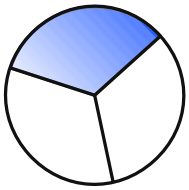




$\frac{\mathbf{2}}{\mathbf{3}}$ of the area of the left square is shaded red. The right square has an additional division of the area, but the amount of the area that is red has stayed the same. We have:

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$$

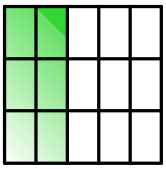


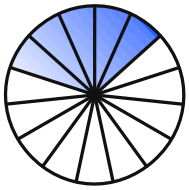


The areas of the square and the circle are the same. $\frac{2}{5}$ of the square is shaded green and $\frac{1}{3}$ of the circle is shaded blue. Which shaded area is larger?

We can divide each fifth of the square into 3 parts. Then the shaded area becomes:

$$\frac{2}{5} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}$$

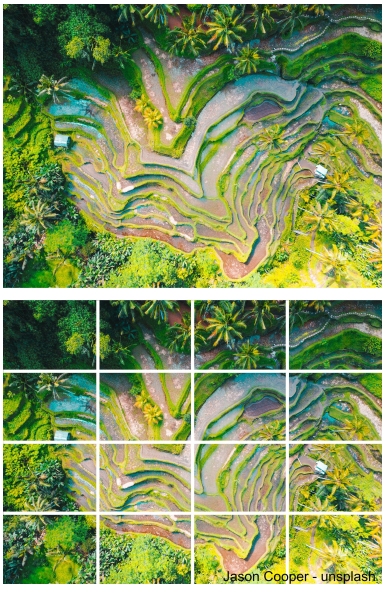




We can also divide each third of the circle into 5 parts. Then the shaded area becomes:

$$\frac{1}{3} = \frac{1 \times 5}{3 \times 5} = \frac{5}{15}$$

Since $\frac{6}{15}$ is greater than $\frac{5}{15}$, the green area is the larger one.



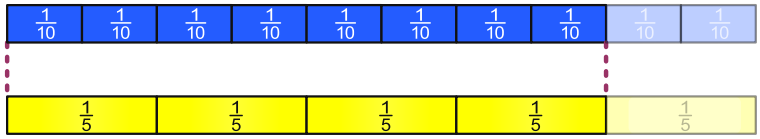
If we have a satellite photo and want to estimate what portion of the area is covered by water, we can divide the photo into equal-sized units and count how many of the units show land and how many show water. Dividing into sixteenths is probably too coarse to make this method useful, so we will have to subdivide the sixteenths into smaller units. Which number we should use to create equivalent fractions depends on how precise we want our estimate to be. We might also consider whether dividing into rectangles is the best solution, or whether dividing into squares might be more useful.

4 Simplifying Fractions

4.1 Definition

To enlarge a fraction, we multiply the numerator and the denominator of this fraction by the same natural number (except 0). This gives us a fraction of equal size. The "reverse" process is also possible.
If the numerator and the denominator of a fraction have a common factor, we can divide both by that factor. This gives us a fraction of equal size.

Let's consider the fraction $\frac{8}{10}$. As we can see, there is a fraction that is equal in size to $\frac{8}{10}$, but with a smaller numerator and denominator, namely $\frac{4}{5}$.



If we divide the denominator 10 by 2, we get 5. $\frac{1}{5}$ is twice as large as $\frac{1}{10}$.
To obtain an equal-sized fraction, we therefore only need half as many parts. So we also divide the numerator 8 by 2 and get 4. We have:

$$\frac{8}{10} = \frac{8 \div 2}{10 \div 2} = \frac{4}{5} .$$

We can express this connection somewhat imprecisely, but very briefly, as follows:

The bigger the parts, the fewer in number.

The process of dividing the numerator and the denominator of a fraction by a common factor of numerator and denominator is called **simplifying**. Simplifying a fraction results in a fraction of equal size.

> If the numerator and the denominator of a fraction are divided by a common factor, a fraction of equal size results. This process is called **simplifying**.

We can also express this connection very briefly:

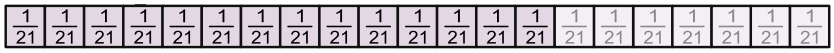
$$\frac{a}{b} = \frac{a \div n}{b \div n}$$

Here, $\frac{\mathbf{a}}{\mathbf{b}}$ stands for any fraction and **n** stands for any common factor of **a** and **b**. To avoid nonsense, we will from now on divide only by such common factors **n** that are greater than **1**.

4.2 Examples

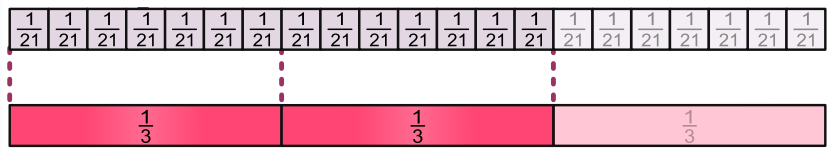
Example 1

We want to simplify the fraction $\frac{14}{21}$ by 7.



If we divide the denominator 21 by 7, we get 3.
$\frac{1}{3}$ is seven times as large as $\frac{1}{21}$.
To get a fraction of equal size,
we therefore only need one seventh as many parts.
If we divide the numerator 14 by 7, we get 2.

$$\frac{14}{21} = \frac{14 \div 7}{21 \div 7} = \frac{2}{3}$$



Example 2

We want to simplify the fraction $\frac{12}{18}$ by 3.

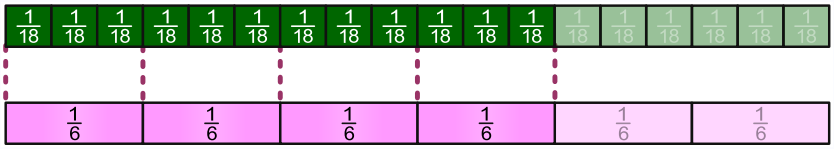
If we divide the denominator 18 by 3, we get 6.

$\frac{1}{6}$ is three times as large as $\frac{1}{18}$.

To get a fraction of equal size, we therefore only need one third as many parts.

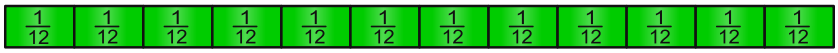
If we divide the numerator 12 by 3, we get 4.

$$\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}\,.$$

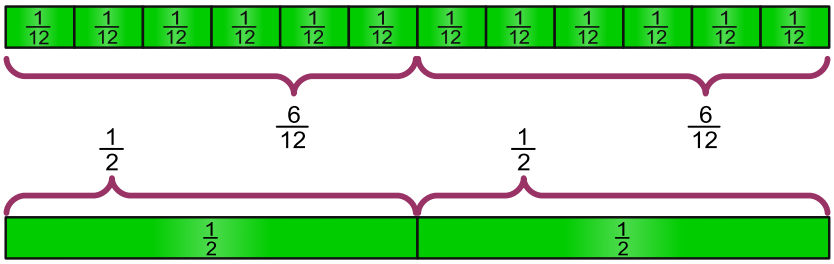


4.3 Simplifying by all divisors of the denominator

For every divisor of a denominator, there are fractions that we can simplify by this divisor. Let's look at an example: Here we have 12 twelfths.

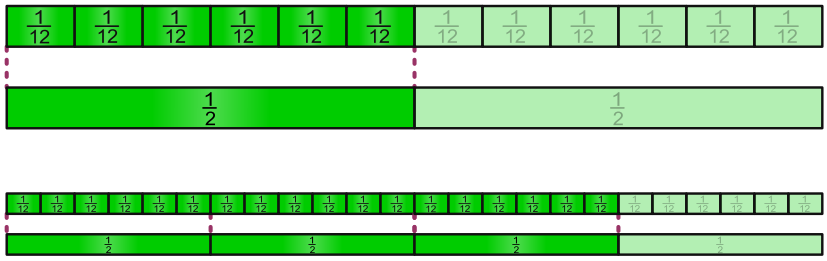


Since 12 is divisible by 6, we can group 6 twelfths into *one* part each. On this fraction strip, there are $12 \div 6 = 2$ such parts. So these are halves.



If a fraction has the denominator 12 and a numerator that is divisible by 6, we can simplify the fraction by 6. For example:

$$\frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2} \quad \text{and} \quad \frac{18}{12} = \frac{18 \div 6}{12 \div 6} = \frac{3}{2}.$$



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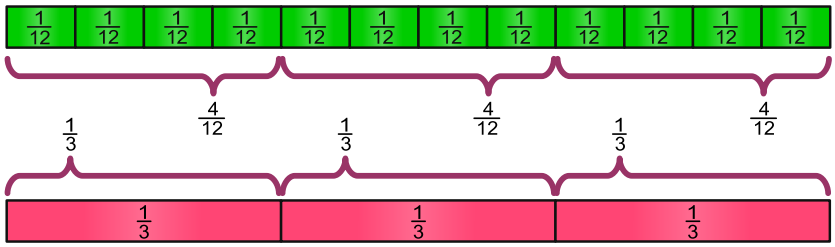
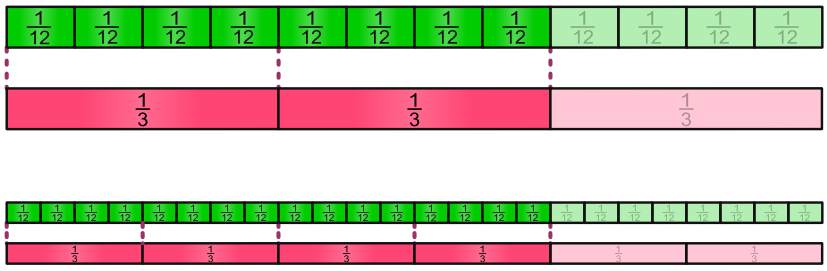
12 is also divisible by 4.
Therefore, we can divide the 12 twelfths into groups of four.
On one whole, we then have 3 equal parts.
Each of these parts is thus equal to $\frac{1}{3}$. So: $\frac{4}{12} = \frac{1}{3}$.



If a fraction has the denominator 12 and a numerator that is divisible by 4, we can simplify the fraction by 4. For example:

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3} \quad \text{and} \quad \frac{16}{12} = \frac{16 \div 4}{12 \div 4} = \frac{4}{3}.$$



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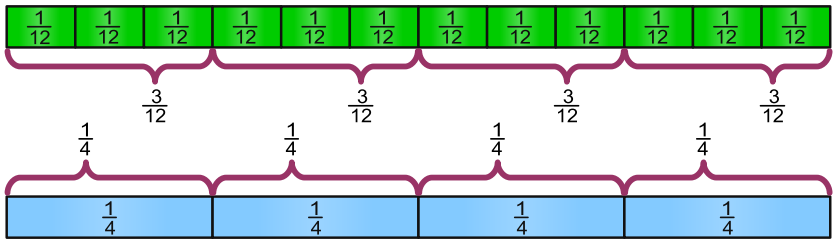
12 is also divisible by 3. Therefore, we can combine 3 twelfths into *one* part and obtain fourths.



If a fraction has the denominator 12 and a numerator that is divisible by 3, we can simplify the fraction by 3. For example:

$$\frac{6}{12} = \frac{6 \div 3}{12 \div 3} = \frac{2}{4} \quad \text{and} \quad \frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}.$$

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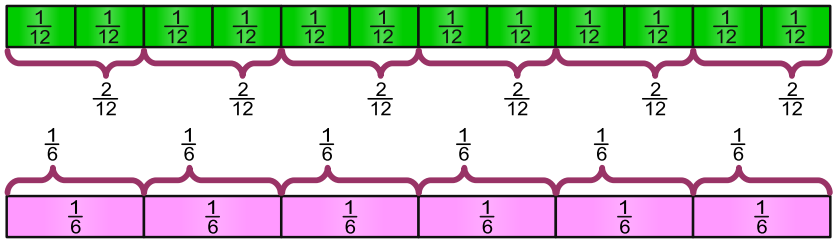
12 is also divisible by 2. Therefore, we can combine 2 twelfths into *one* part and obtain sixths.



If a fraction has the denominator 12 and a numerator that is divisible by 2, we can simplify the fraction by 2. For example:

$$\frac{6}{12} = \frac{6 \div 2}{12 \div 2} = \frac{3}{6} \quad \text{and} \quad \frac{28}{12} = \frac{28 \div 2}{12 \div 2} = \frac{14}{6}.$$

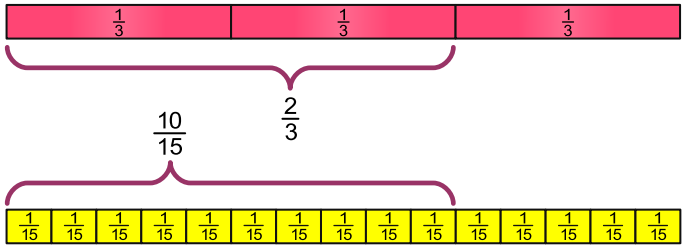
4.4 Simplifying as the inverse of expanding

We can also think of simplifying as the inverse of expanding. Expanding means, for example: We divide each third of $\frac{2}{3}$ into 5 equal parts. Each of these parts then has one fifth the size of a third. Therefore, we need 5 times as many parts for a fraction of equal size.

So: $\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$.

Now we can go the other way and group every 5 of the 10 fifteenths into larger parts. These parts are then 5 times as large as the previous ones, and therefore we only need $10 \div 5 = 2$ parts for a fraction of equal size.

So: $\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$.



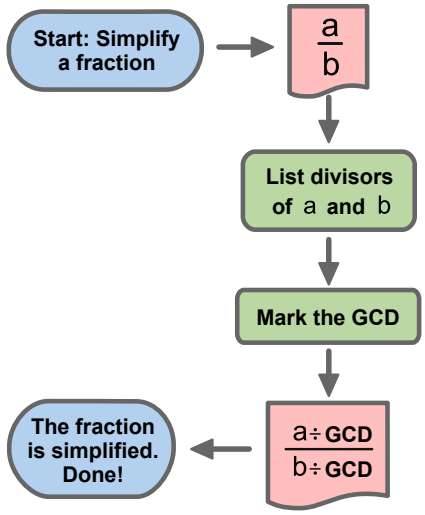
4.5 Simplifying a Fraction

Calculating with simplified fractions is usually easier than calculating with unsimplified fractions. We usually simplify a fraction using the greatest common divisor of the numerator and the denominator.
If we are given a fraction, we can proceed as follows:

1. Write down all divisors of the numerator.
2. Write down all divisors of the denominator.
3. Mark the greatest common divisor GCD of numerator and denominator.
4. Divide numerator and denominator by the greatest common divisor GCD.

From now on, whenever we talk about simplifying a fraction, we always mean simplifying a fraction using the GCD.

We can also represent this procedure as a flowchart.



Let's look at some examples:

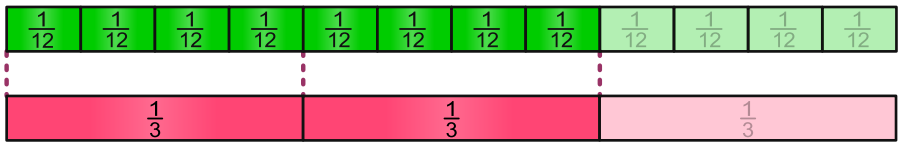
4.6 Examples

Example 1

The fraction $\frac{8}{12}$ has numerator 8 and denominator 12.
The divisors of 8 are: $1; 2; 4; 8$.
The divisors of 12 are: $1; 2; 3; 4; 6; 12$.
We have marked the GCD in red.
Now we divide the numerator and denominator by the GCD.

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

So if we simplify $\frac{8}{12}$, we get $\frac{2}{3}$.

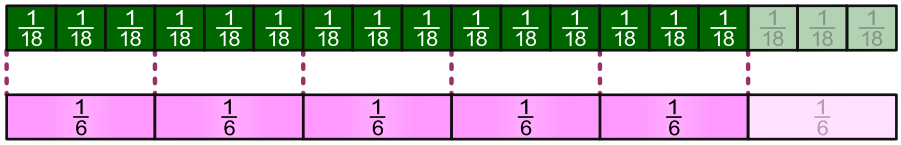


Example 2

The fraction $\frac{15}{18}$ has numerator 15 and denominator 18.
The divisors of 15 are: $1; 3; 5; 15$.
The divisors of 18 are: $1; 2; 3; 6; 9; 18$.
We have marked the GCD in red.
Now we divide the numerator and denominator by the GCD.

$$\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$$

So if we simplify $\frac{15}{18}$, we get $\frac{5}{6}$.

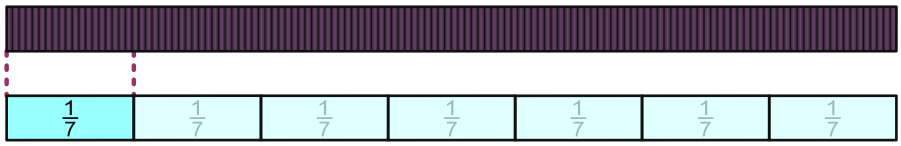


Example 3

The fraction $\frac{108}{756}$ has numerator 108 and denominator 756.
The divisors of 108 are: $1; 2; 3; 4; 6; 9; 12; 18; 27; 36; 54; 108$.
The divisors of 756 are: $1; 2; 3; 4; 6; 7; 9; 12; 14; 18; 21; 27; 28; 36; 42; 54; 63; 84; 108; 126; 189; 252; 378; 756$.
We have marked the GCD in red.
Now we divide numerator and denominator by the GCD.

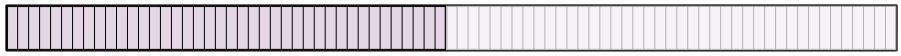
$$\frac{108}{756} = \frac{108 : 108}{756 : 108} = \frac{1}{7}$$

So, if we reduce the fraction $\frac{108}{756}$, we get $\frac{1}{7}$.



Example 4

The fraction $\frac{40}{81}$ has numerator 40 and denominator 81.
The divisors of 40 are: $1; 2; 4; 5; 10; 20; 40$.
The divisors of 81 are: $1; 3; 9; 27; 81$.
We have marked the GCD in red.
Since the greatest common divisor of numerator and denominator is 1, this fraction cannot be reduced in a meaningful way.

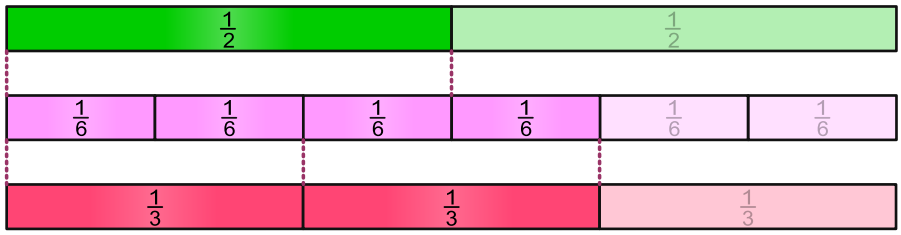


5 Finding the Least Common Denominator

5.1 Making Fractions Have the Same Denominator

We can compare the sizes of fractions especially well, or add and subtract them more easily, when they have the same denominator.
If two fractions have different denominators, we can rewrite them as equivalent fractions with the same denominator.
Let's look at this idea using fraction strips:

We have the fractions $\frac{1}{2}$ and $\frac{2}{3}$. We want to rewrite both fractions as equivalent fractions with the same denominator. Sixths are a good option for this.



We find an equivalent fraction for $\frac{1}{2}$ by multiplying the numerator and the denominator by 3:

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$

We find an equivalent fraction for $\frac{2}{3}$ by multiplying the numerator and the denominator by 3:

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}$$

Both resulting fractions now have the same denominator 6.

Vocabulary

The process of changing fractions so that they have the same denominator is called:

- **finding a common denominator**
- **rewriting with a common denominator**
- **making the denominators the same**
- you may also hear: **bringing to a common denominator**

Fractions with the same denominator are called **like fractions**.
Fractions with different denominators are called **unlike fractions**.

Here is an example without any special features:

Example

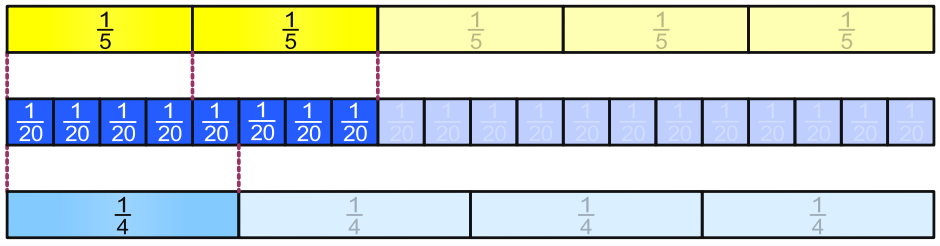
We want to write the two fractions $\frac{2}{5}$ and $\frac{1}{4}$ with the same denominator.
To do this, we look for a common multiple of the denominators 5 and 4, and we choose 20.
We multiply the numerator and denominator of $\frac{2}{5}$ by 4:

$$\frac{2}{5} = \frac{2 \times 4}{5 \times 4} = \frac{8}{20}$$

We multiply the numerator and denominator of $\frac{1}{4}$ by 5:

$$\frac{1}{4} = \frac{1 \times 5}{4 \times 5} = \frac{5}{20}$$

The two equivalent fractions now have the same denominator, 20.



5.2 Simplify first

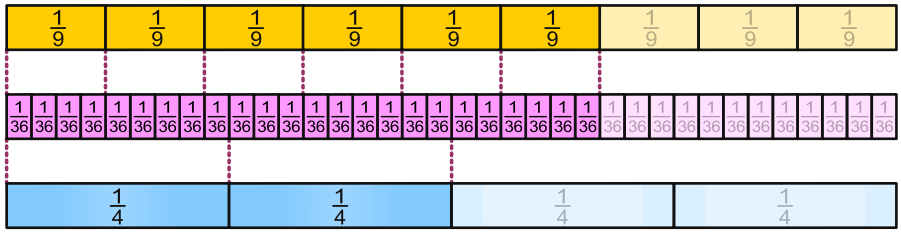
If we simplify the given fractions first before rewriting them with a common denominator, we usually get a smaller common denominator. This is especially practical when we want to add or subtract the rewritten fractions. Therefore, we will get used to simplifying fractions before making them like denominators.

Let us look at the fractions $\frac{\mathbf{6}}{\mathbf{9}}$ and $\frac{\mathbf{2}}{\mathbf{4}}$.

We can rewrite the fractions – without simplifying – by multiplying numerator and denominator of each fraction by the denominator of the other fraction. That is:

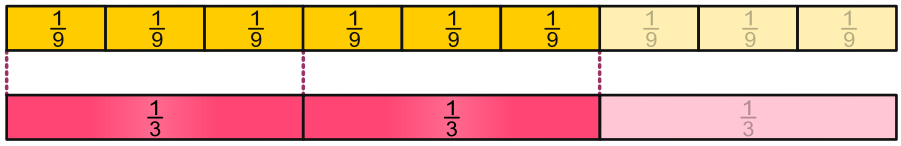
$$\frac{6}{9} = \frac{6 \times 4}{9 \times 4} = \frac{24}{36} \quad \text{and} \quad \frac{2}{4} = \frac{2 \times 9}{4 \times 9} = \frac{18}{36} \, .$$

The common denominator of the rewritten fractions is 36.

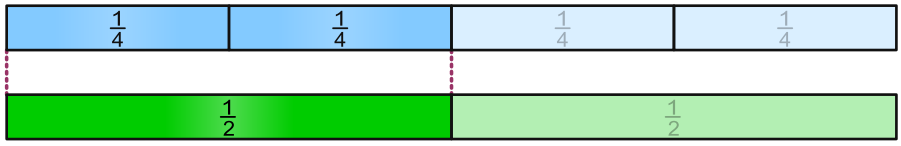


Now the fractions should be simplified first:

$$\frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$$



$$\text{and} \quad \frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2} \, .$$



Now we rewrite the simplified fractions with a common denominator.
A common multiple of the denominators 3 and 2 is 6.

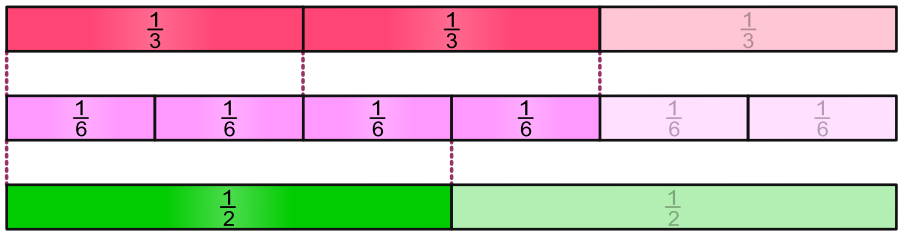
We multiply the numerator and denominator of $\frac{2}{3}$ by 2

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}$$

We multiply the numerator and denominator of $\frac{1}{2}$ by 3

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$

The two rewritten fractions now have the same denominator, 6.



As we can see, we get a smaller common denominator if we simplify the fractions before making them like denominators.

5.3 Rewrite with the LCM

When we rewrite two fractions with a common denominator, for example to add or subtract them, we do not want to use unnecessarily large denominators. That's why we don't rewrite the fractions with just *any* common denominator, but with the *least* common denominator. The least common denominator of two (or more) fractions is the **l**east **c**ommon **m**ultiple of all denominators (abbreviated as **LCM**).
We can see how to systematically find the least common denominator of two fractions in this example:

1. Which fractions?

We want to rewrite the fractions $\frac{3}{4}$ and $\frac{5}{6}$ with the least common denominator.

2. Simplify?

First, we check whether the fractions can be simplified and realize that this is not the case.

3. Find the LCM

Now we list the multiples of the denominators:

Multiples of 4 are: $4; 8; 12; 16; 20; 24; ...$

Multiples of 6 are: $6; 12; 18; 24; ...$

We have marked the least common multiple of the two denominators in red.

4. Divide LCM by denominator

To find out by which number we need to rewrite $\frac{3}{4}$ in order to get the least common denominator 12, we divide the least common multiple – the LCM – by the denominator of the fraction. So:

$$\frac{12}{4} = 3$$

To find out how to rewrite $\frac{5}{6}$ with the least common denominator, we also divide the LCM 12 by this denominator:

$$\frac{12}{6} = 2$$

5. Rewrite

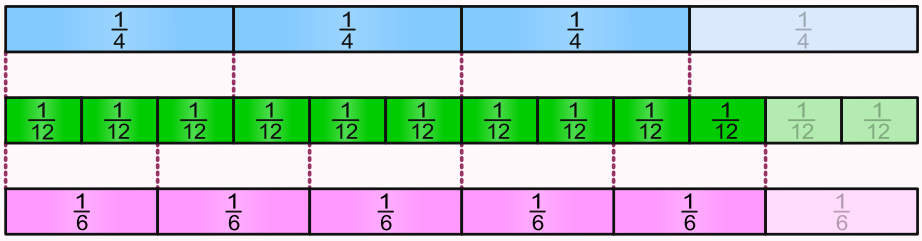
We rewrite $\frac{3}{4}$ by multiplying numerator and denominator by 3

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

And we rewrite $\frac{5}{6}$ by multiplying numerator and denominator by 2

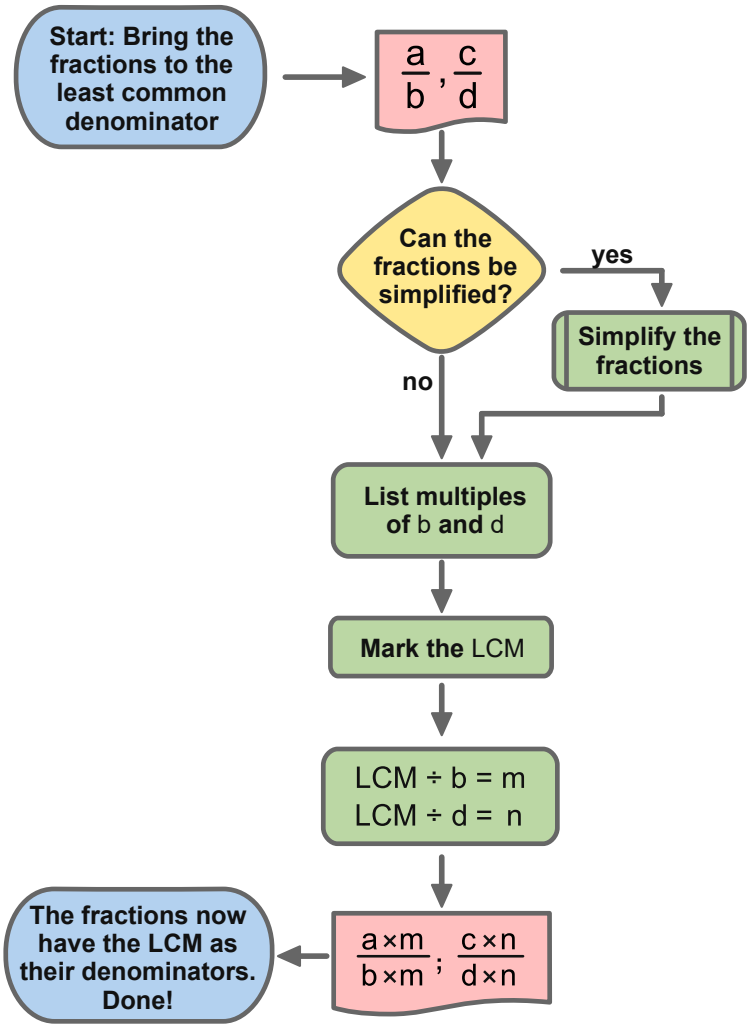
$$\frac{5}{6} = \frac{5 \times 2}{6 \times 2} = \frac{10}{12}$$

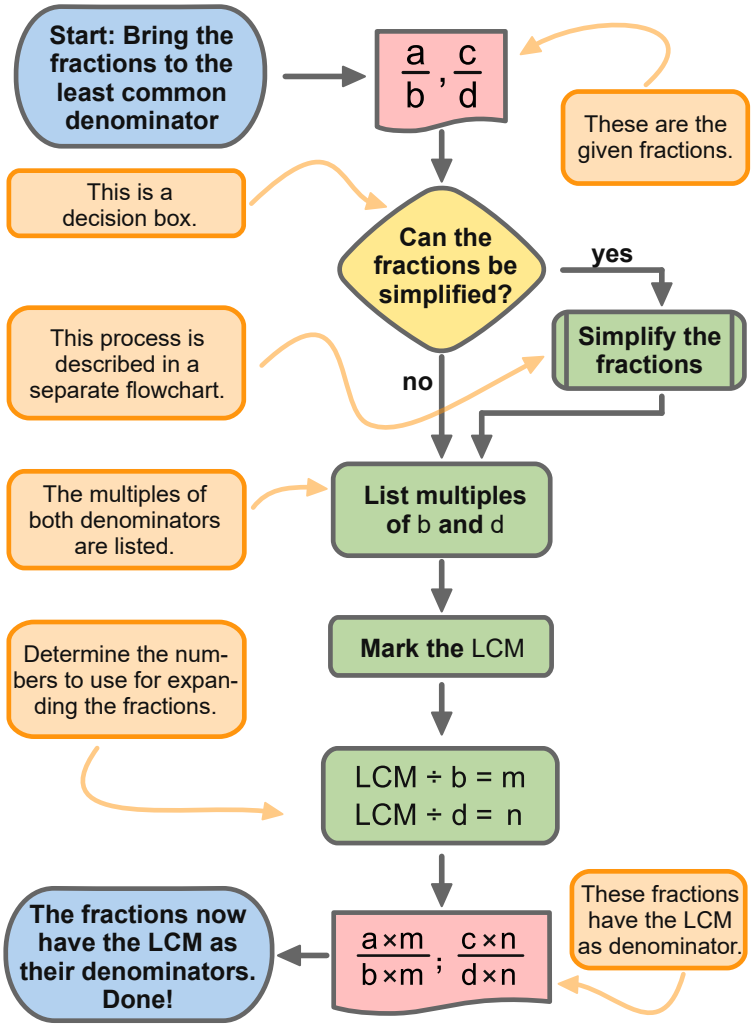
We have now rewritten both fractions with the least common denominator 12.



5.4 Flowchart

The process of rewriting fractions with the least common denominator can also be described with a flowchart. Below is the flowchart without comments. On the next page, the same diagram is shown with explanations.




Start: Bring the fractions to the least common denominator
a/b , c/d
These are the given fractions.
This is a decision box.
Can the fractions be simplified?
yes
Simplify the fractions
This process is described in a separate flowchart.
no
The multiples of both denominators are listed.
List multiples of b and d
Mark the LCM
Determine the numbers to use for expanding the fractions.
LCM ÷ b = m
LCM ÷ d = n
The fractions now have the LCM as their denominators. Done!
a×m / b×m ; c×n / d×n
These fractions have the LCM as denominator.

5.5 Examples

Let's look at a few examples.
Example 1: without any special features

1. What fractions?

We want to rewrite the fractions $\frac{6}{7}$ and $\frac{4}{5}$ with the least common denominator.

2. Simplify?

We first check whether the fractions $\frac{6}{7}$ and $\frac{4}{5}$ can be simplified and see that they cannot be simplified further.

3. Find the least common multiple

We now list the multiples of the denominators:
Multiples of 7 are: 7, 14, 21, 28, 35, 42, 49, ...
Multiples of 5 are: 5, 10, 15, 20, 25, 30, 35, 40, 45, ...
The least common multiple of both denominators is highlighted in red.

4. Divide the least common denominator by each denominator

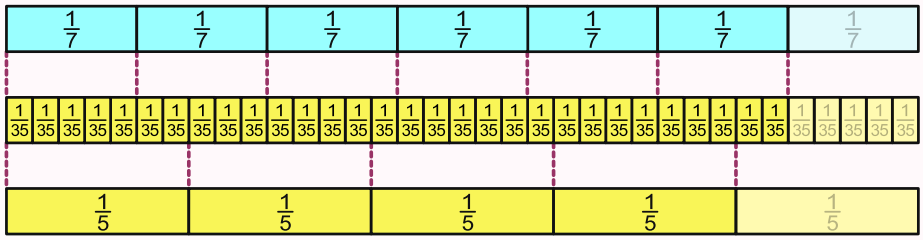
$$35 \div 7 = 5 \quad \text{and} \quad 35 \div 5 = 7$$

5. Rewrite with the least common denominator

We rewrite $\frac{6}{7}$ with denominator 35: $\frac{6}{7} = \frac{6 \times 5}{7 \times 5} = \frac{30}{35}$

And we rewrite $\frac{4}{5}$ with denominator 35: $\frac{4}{5} = \frac{4 \times 7}{5 \times 7} = \frac{28}{35}$

We have now rewritten both fractions with the least common denominator 35.



Example 2:
One denominator is a multiple of the other.
We want to rewrite $\frac{2}{3}$ and $\frac{7}{9}$ with the least common denominator. Since $3 \times 3 = 9$, the denominator of $\frac{7}{9}$ is a multiple of the denominator of $\frac{2}{3}$. So 9 is the least common denominator. To rewrite both fractions with this denominator, we only need to rewrite the first fraction with denominator 9. But we can also follow our usual method step by step.

1. What fractions?

We want to rewrite the fractions $\frac{2}{3}$ and $\frac{7}{9}$ with the least common denominator.

2. Simplify?

We first check whether the fractions $\frac{2}{3}$ and $\frac{7}{9}$ can be simplified and see that they cannot be simplified further.

3. Find the least common multiple

We now list the multiples of the denominators:
Multiples of 3: 3, 6, 9, 12, 15, ...
Multiples of 9: 9, 18, 27, ...
The *least common multiple* of both denominators is highlighted in red.

4. Divide the least common denominator by each denominator

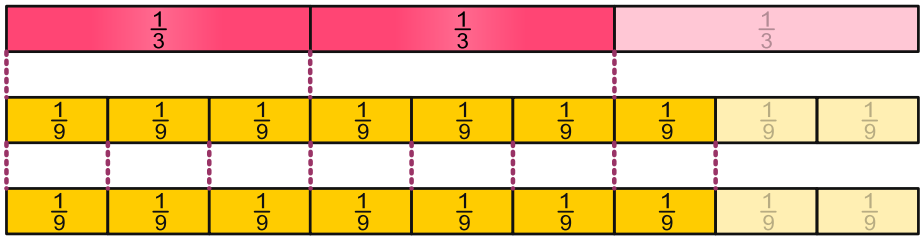
$$9 \div 3 = 3 \quad \text{and} \quad 9 \div 9 = 1$$

5. Rewrite with the least common denominator

We rewrite $\frac{2}{3}$ with denominator 9: $\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}$

And we rewrite $\frac{7}{9}$ with denominator 9: $\frac{7}{9} = \frac{7 \times 1}{9 \times 1} = \frac{7}{9}$

We have now rewritten both fractions with the least common denominator 9.



Example 3:
The denominators have common factors.
We want to rewrite $\frac{3}{8}$ and $\frac{5}{6}$ with the least common denominator. Since both denominators have common factors, the least common denominator is smaller than the product of the two denominators.

1. What fractions?

We want to rewrite the fractions $\frac{3}{8}$ and $\frac{5}{6}$ with the least common denominator.

2. Simplify?

We first check whether the fractions $\frac{3}{8}$ and $\frac{5}{6}$ can be simplified and find that they cannot.

3. Find the least common multiple

We now list the multiples of the denominators:
Multiples of 8: 8, 16, 24, 32, 40, ...
Multiples of 6: 6, 12, 18, 24, 30, 36, ...
The *least common multiple* of both denominators is highlighted in red.

4. Divide the least common denominator by each denominator

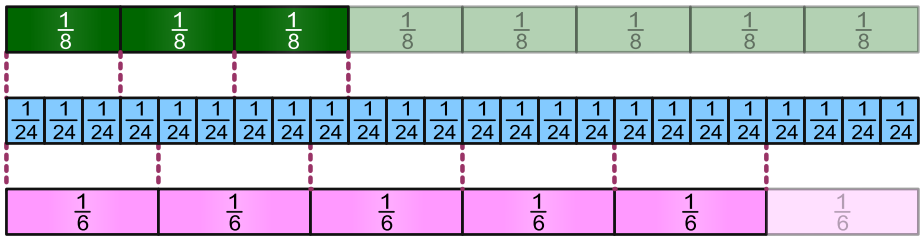
$$24 \div 8 = 3 \quad \text{and} \quad 24 \div 6 = 4$$

5. Rewrite with the least common denominator

We rewrite $\frac{3}{8}$ with denominator 24:
$$\frac{3}{8} = \frac{3 \times 3}{8 \times 3} = \frac{9}{24}$$

And we rewrite $\frac{5}{6}$ with denominator 24:
$$\frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24}$$

We have now rewritten both fractions with the least common denominator 24.



Example 4: One of the fractions can be simplified.
We want to rewrite $\frac{10}{15}$ and $\frac{7}{10}$ with the least common denominator. In this case, simplifying the first fraction does not change the least common denominator, because the LCM of 15 and 10 is 30, which is the same as the LCM of 3 and 10.

1. What fractions?

We want to rewrite the fractions $\frac{10}{15}$ and $\frac{7}{10}$ with the least common denominator.

2. Simplify?

We simplify the first fraction: $\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$
From now on, we look for the least common denominator of $\frac{2}{3}$ and $\frac{7}{10}$.

3. Find the least common multiple

Now we list the multiples of the denominators:
Multiples of 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, ...
Multiples of 10: 10, 20, 30, 40, ...
The *least common multiple* of both denominators is highlighted in red.

4. Divide the least common denominator by each denominator

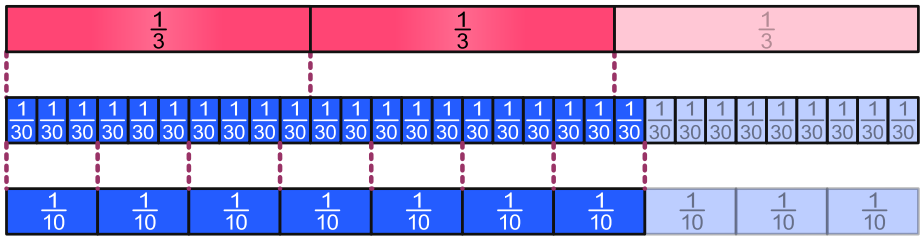
$$30 \div 3 = 10 \quad \text{and} \quad 30 \div 10 = 3$$

5. Rewrite with the least common denominator

We rewrite $\frac{2}{3}$ with denominator 30: $\frac{2}{3} = \frac{2 \times 10}{3 \times 10} = \frac{20}{30}$

And we rewrite $\frac{7}{10}$ with denominator 30: $\frac{7}{10} = \frac{7 \times 3}{10 \times 3} = \frac{21}{30}$

We have now rewritten both fractions with the least common denominator 30.



Example 5: Both fractions can be simplified.
We want to rewrite $\frac{7}{35}$ and $\frac{55}{66}$ with the least common denominator. If we didn't simplify and just multiplied the denominators, the common denominator would be 2310, which would be rather inconvenient for further calculations.

1. What fractions?

We want to rewrite the fractions $\frac{7}{35}$ and $\frac{55}{66}$ with the least common denominator.

2. Simplify?

We simplify the fractions: $\frac{7}{35} = \frac{7 \div 7}{35 \div 7} = \frac{1}{5}$; $\frac{55}{66} = \frac{55 \div 11}{66 \div 11} = \frac{5}{6}$

Now we look for the least common denominator of $\frac{1}{5}$ and $\frac{5}{6}$.

3. Find the least common multiple

Now we list the multiples of the denominators:
Multiples of 5: 5, 10, 15, 20, 25, 30, 35, ...
Multiples of 6: 6, 12, 18, 24, 30, 36, ...
The *least common multiple* of both denominators is highlighted in red.

4. Divide the least common denominator by each denominator

$30 \div 5 = 6 \quad \text{and} \quad 30 \div 6 = 5$

5. Rewrite with the least common denominator

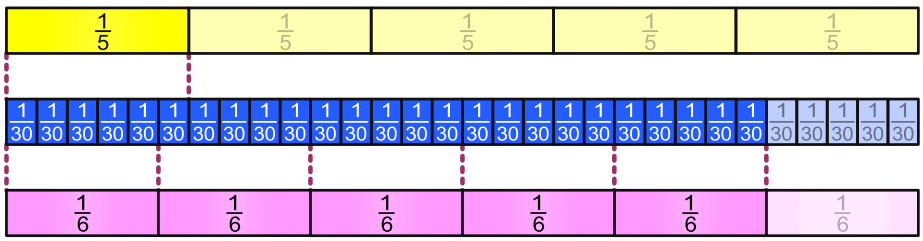
We rewrite $\frac{1}{5}$ with denominator 30:

$$\frac{1}{5} = \frac{1 \times 6}{5 \times 6} = \frac{6}{30}$$

And we rewrite $\frac{5}{6}$ with denominator 30:

$$\frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$$

We have now rewritten both fractions with the least common denominator 30.



Example 6: The denominators of the simplified fractions share common factors. We want to rewrite $\frac{28}{40}$ and $\frac{39}{75}$ with the least common denominator. The denominators of the simplified fractions share common factors, so the least common multiple is smaller than the product of the two denominators.

1. What fractions?

We want to rewrite the fractions $\frac{28}{40}$ and $\frac{39}{75}$ with the least common denominator.

2. Simplify?

We simplify the fractions: $\frac{28}{40} = \frac{28 \div 4}{40 \div 4} = \frac{7}{10}$; $\frac{39}{75} = \frac{39 \div 3}{75 \div 3} = \frac{13}{25}$

Now we look for the least common denominator of $\frac{7}{10}$ and $\frac{13}{25}$.

3. Find the least common multiple

Now we list the multiples of the denominators:
Multiples of 10: 10, 20, 30, 40, 50, 60, ...
Multiples of 25: 25, 50, 75, ...
The *least common multiple* of both denominators is highlighted in red.

4. Divide the least common denominator by each denominator

$$50 \div 10 = 5 \quad \text{and} \quad 50 \div 25 = 2$$

5. Rewrite with the least common denominator

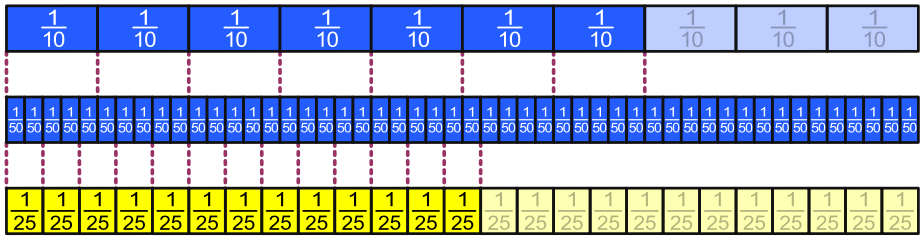
We rewrite $\frac{7}{10}$ with denominator 50:

$$\frac{7}{10} = \frac{7 \times 5}{10 \times 5} = \frac{35}{50}$$

And we rewrite $\frac{13}{25}$ with denominator 50:

$$\frac{13}{25} = \frac{13 \times 2}{25 \times 2} = \frac{26}{50}$$

We have now rewritten both fractions with the least common denominator 50.

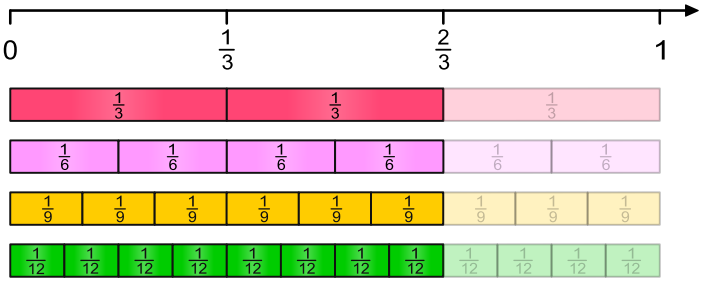


6 Comparing Fractions

6.1 Equivalent Fractions

For every fraction we find on the number line, there are other fractions that are equally large. For example: $\frac{2}{3} = \frac{4}{6} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} =$ and so on.

If we write an equivalent fraction for $\frac{2}{3}$ by multiplying numerator and denominator by 7, 8, 9, ..., we obtain other fractions of the same value.

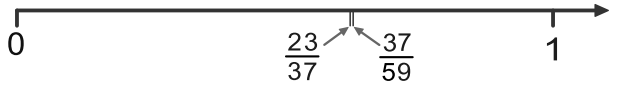


We can imagine it like this on the number line.

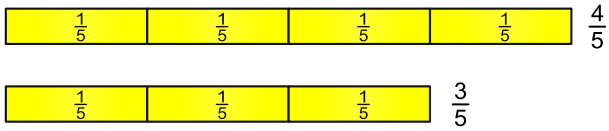
At every point where a natural number is located, there are also many fractions. $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \frac{6}{6} =$ and so on.

6.2 Comparing Fractions

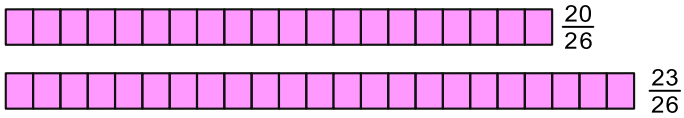
We can always tell which of two natural numbers is greater. For example, we immediately know that 89 is greater than 78 without having to think about it. But when we are given the fractions $\frac{8}{9}$ and $\frac{7}{8}$, it is not so obvious.
If we want to know whether two fractions are equal or which one is greater, we could plot the fractions on the number line. If both fractions are located at the same point, they are equal. If one of the fractions is to the right of the other, then the one on the right is greater and the one on the left is smaller.
But this method can be very tedious. If the fractions are close together, the difference may be hard to see, as with $\frac{23}{37}$ and $\frac{37}{59}$.



Therefore, we need a method to quickly and reliably compare two fractions. One possible solution is to bring the fractions to the same denominator and then compare the numerators.
Explanation: When two fractions have the same denominator, we call them **like fractions**.
Like fractions can be compared by comparing their numerators. For example: $\frac{4}{5}$ is greater than $\frac{3}{5}$ — written as $\frac{4}{5} > \frac{3}{5}$ (read as "four fifths greater than three fifths") — because 4 is greater than 3.



Even if the denominators are quite large, we can still compare the fractions easily. $\frac{20}{26} < \frac{23}{26}$ (read as: "twenty twenty-sixths is less than twenty-three twenty-sixths").



If we want to compare two like fractions, the size of the denominators does not matter at all. As long as the *numerator* of one fraction is greater than that of the other, the *entire fraction* is greater than the other.

$$\frac{20}{\text{any denominator}} < \frac{23}{\text{any denominator}}$$

Let's summarize our method:

How to compare fractions:
1) Bring the fractions to a common denominator.
2) Compare the numerators: The fraction with the largest numerator is the greatest, the fraction with the smallest numerator is the smallest.

Example 1

We want to compare the fractions $\frac{\mathbf{13}}{\mathbf{31}}$ and $\frac{\mathbf{17}}{\mathbf{31}}$.

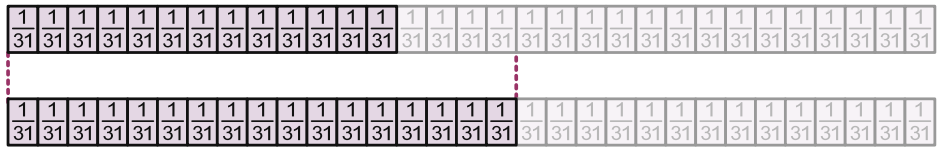
Since both fractions have the same denominator, we do not need to bring the fractions to a common denominator.

We have $\frac{\mathbf{13}}{\mathbf{31}}$ less than $\frac{\mathbf{17}}{\mathbf{31}}$.

We can write this more concisely as:

$$\frac{13}{31} < \frac{17}{31}$$

Read as: Thirteen thirty-firsts is less than seventeen thirty-firsts.



Example 2

We want to compare the fractions $\frac{5}{7}$ and $\frac{2}{3}$.

The least common denominator is the LCM of 7 and 3, which is 21. We convert:

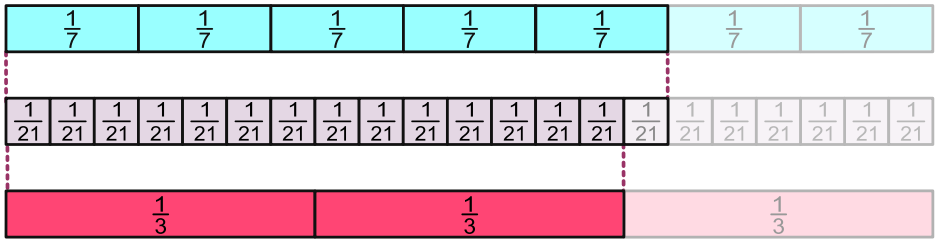
$$\frac{5}{7} = \frac{5 \times 3}{7 \times 3} = \frac{15}{21} \quad ; \quad \frac{2}{3} = \frac{2 \times 7}{3 \times 7} = \frac{14}{21}$$

Since $\frac{15}{21}$ is greater than $\frac{14}{21}$, it follows that $\frac{5}{7}$ is greater than $\frac{2}{3}$.

We can write this more concisely as:

$$\frac{15}{21} = \frac{5}{7} > \frac{2}{3} = \frac{14}{21}$$

Read as: Fifteen twenty-firsts equals five sevenths is greater than two thirds equals fourteen twenty-firsts.



Example 3

We want to compare the fractions $\frac{19}{24}$ and $\frac{13}{16}$.

The least common denominator is the LCM of 24 and 16, which is 48.

We convert:

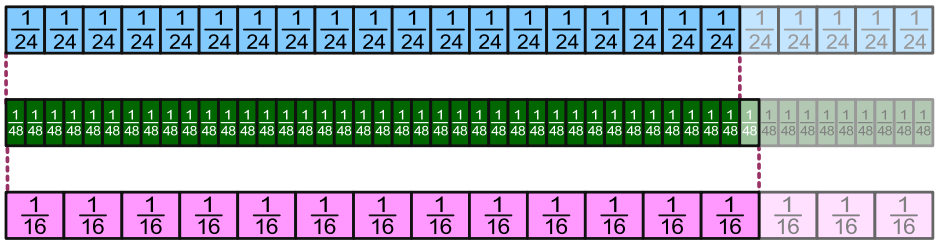
$$\frac{19}{24} = \frac{19 \times 2}{24 \times 2} = \frac{38}{48} \quad ; \quad \frac{13}{16} = \frac{13 \times 3}{16 \times 3} = \frac{39}{48}$$

Because $\frac{38}{48}$ is less than $\frac{39}{48}$, $\frac{19}{24}$ is less than $\frac{13}{16}$.

We can write this more concisely as:

$$\frac{38}{48} = \frac{19}{24} < \frac{13}{16} = \frac{39}{48}$$

Read as: Thirty-eight forty-eighths equals nineteen twenty-fourths is less than thirteen sixteenths equals thirty-nine forty-eighths.



Example 4

We want to compare the fractions $\frac{5}{8}$ and $\frac{5}{9}$.

The least common denominator is the LCM of 8 and 9, which is 72. We convert:

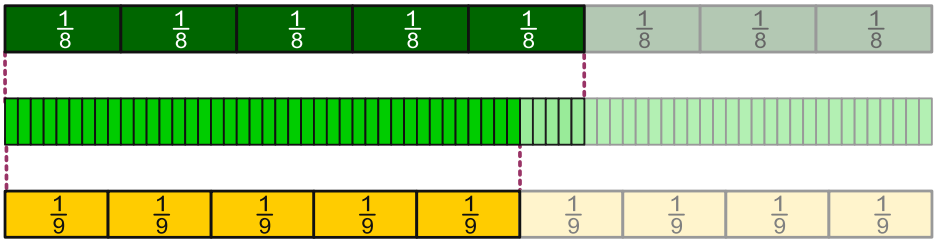
$$\frac{5}{8} = \frac{5 \times 9}{8 \times 9} = \frac{45}{72} \quad ; \quad \frac{5}{9} = \frac{5 \times 8}{9 \times 8} = \frac{40}{72}$$

Because $\frac{45}{72}$ is greater than $\frac{40}{72}$, $\frac{5}{8}$ is greater than $\frac{5}{9}$.

We can write this more concisely as:

$$\frac{45}{72} = \frac{5}{8} > \frac{5}{9} = \frac{40}{72}$$

Read as: Forty-five seventy-seconds equals five eighths is greater than five ninths equals forty seventy-seconds.



We could also consider the following in this case:

The smaller the denominator of a fraction, the larger the parts.

Thus, $\frac{1}{8}$ is greater than $\frac{1}{9}$.

Therefore, $\frac{5}{8}$ is also greater than $\frac{5}{9}$.

In general: If two fractions have the same numerator, the fraction with the smaller denominator is the larger one.

Example 5

We want to compare the fractions $\frac{6}{7}$ and $\frac{5}{6}$.

The least common denominator is the LCM of 7 and 6, which is 42. We convert:

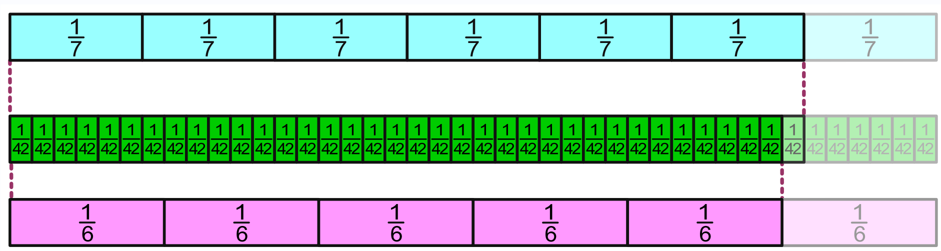
$$\frac{6}{7} = \frac{6 \cdot 6}{7 \cdot 6} = \frac{36}{42} \quad ; \quad \frac{5}{6} = \frac{5 \cdot 7}{6 \cdot 7} = \frac{35}{42}$$

Since $\frac{36}{42}$ is greater than $\frac{35}{42}$, $\frac{6}{7}$ is greater than $\frac{5}{6}$.

We can write this more concisely as:

$$\frac{36}{42} = \frac{6}{7} > \frac{5}{6} = \frac{35}{42}$$

Read as: Thirty-six forty-seconds equals six sevenths is greater than five sixths equals thirty-five forty-seconds.



In this case, we could also have reasoned as follows:

To reach one whole from $\frac{6}{7}$ we are missing $\frac{1}{7}$.

To reach one whole from $\frac{5}{6}$ we are missing $\frac{1}{6}$.

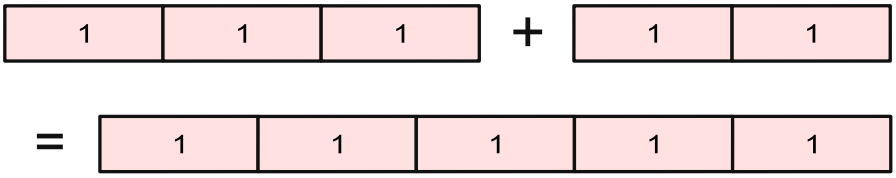
Since $\frac{1}{6}$ is greater than $\frac{1}{7}$, more is missing from $\frac{5}{6}$ than from $\frac{6}{7}$ to make one whole.

Therefore, $\frac{6}{7}$ is greater than $\frac{5}{6}$.

7 Adding Fractions

7.1 Adding Like Fractions

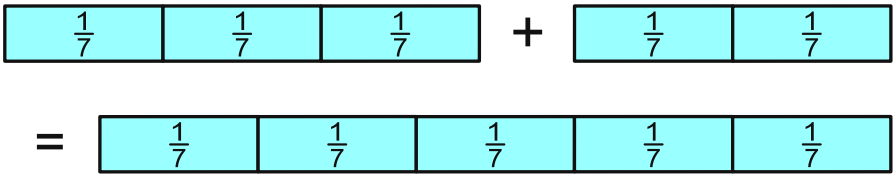
We know how to add natural numbers. For example, we calculate: $3 + 2 = 5$.
We can imagine this using number strips:



We want to add fractions in a similar way. For example, we want the following to be valid:

$$\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

.
Here is how this looks using fraction strips:



As we can see, we can add like fractions by adding the numerators and keeping the common denominator. Therefore, we define:

> Like fractions are added
> by adding the numerators.
> The denominator remains the same.

We can write this relationship also as a formula:

$$\frac{\mathbf{a}}{\mathbf{c}} + \frac{\mathbf{b}}{\mathbf{c}} = \frac{\mathbf{a} + \mathbf{b}}{\mathbf{c}}$$

Whenever we replace the variables with numbers, we obtain a true equation.
For example:
If we replace **a** with **2** , **b** with **5** , and **c** with **3**, we obtain the true equation:

$$\frac{2}{3} + \frac{5}{3} = \frac{2+5}{3} = \frac{7}{3}$$

We can also replace **a** with **7**, **b** with **1**, and **c** with **9**. Again, we obtain a true equation:

$$\frac{7}{9} + \frac{1}{9} = \frac{7+1}{9} = \frac{8}{9}$$

Adding unlike fractions

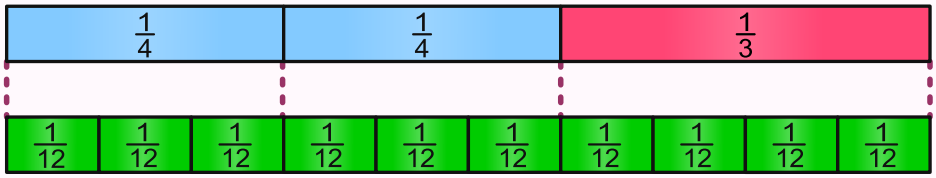
Problem: When we want to add unlike fractions, we have to take a different approach, since the denominators cannot remain the same — they are different. For example:

Solution: We convert the fractions to a common denominator and then add the equivalent fractions. Here is the reasoning:

The result of, for example, $\frac{2}{4} + \frac{1}{3}$ should represent the total length of the two fraction strips.

We arrive at the same total length if we replace $\frac{2}{4}$ and $\frac{1}{3}$ with other fractions of the same size.

For example, we can replace $\frac{2}{4}$ with $\frac{6}{12}$ and $\frac{1}{3}$ with $\frac{4}{12}$.



Since $\frac{6}{12}$ and $\frac{4}{12}$ have the same denominator, we can add the two fractions by adding the numerators and keeping the denominators the same.

So: $\frac{6}{12} + \frac{4}{12} = \frac{10}{12}$.

When adding fractions, there are three more important points to consider:

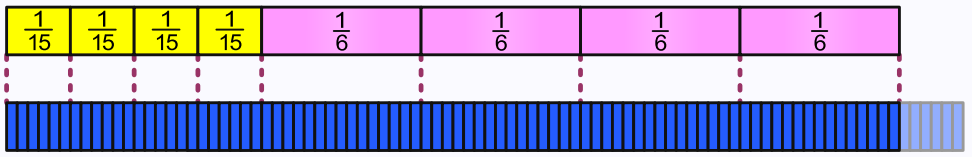
1. Least Common Denominator

If we want to add two unlike fractions, we could find a common denominator by multiplying the denominators of the two fractions. However, this may lead to unnecessarily large numbers. Therefore, we usually rewrite fractions withthe least common multiple of the denominators — that is, the least common denominator.

Let's look at an example:

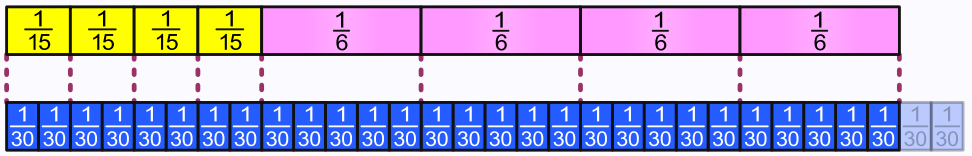
If we want to add the fractions $\frac{4}{15}$ and $\frac{4}{6}$, we could multiply each fraction by the denominator of the other fraction. Then we would have to work with ninetyths:

$$\frac{4}{15}+\frac{4}{6}=\frac{4\times 6}{15\times 6}+\frac{4\times 15}{6\times 15}=\frac{24}{90}+\frac{60}{90}=\frac{84}{90}$$



However, since the least common multiple (LCM) of 15 and 6 is 30, it is sufficient to rewrite both fractions using thirtieths:

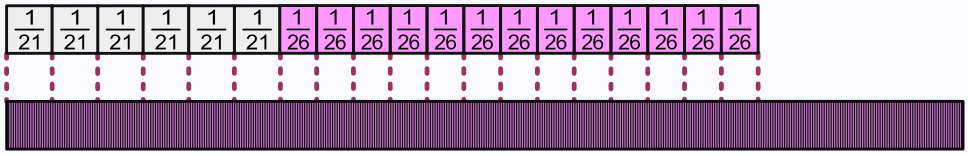
$$\frac{4}{15}+\frac{4}{6}=\frac{4\times 2}{15\times 2}+\frac{4\times 5}{6\times 5}=\frac{8}{30}+\frac{20}{30}=\frac{28}{30}$$



2. At the Beginning: Simplify

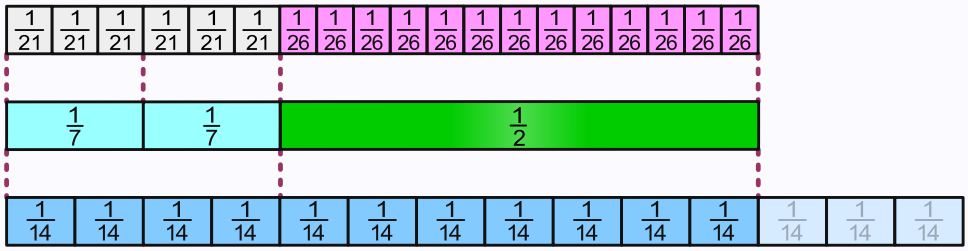
To avoid working with unnecessarily large numbers, we first simplify both fractions as much as possible before adding them.
Let's look at the benefit with an example:
If we want to add the fractions $\frac{\mathbf{6}}{\mathbf{21}}$ and $\frac{\mathbf{13}}{\mathbf{26}}$, we could rewrite both fractions using
the LCM of 21 and 26. Then we would have to work with five-hundred-forty-sixths:

$$\frac{6}{21}+\frac{13}{26}=\frac{6\times 26}{21\times 26}+\frac{13\times 21}{26\times 21}=\frac{156}{546}+\frac{273}{546}=\frac{429}{546}$$



However, if we simplify first, we end up with a much smaller least common denominator.

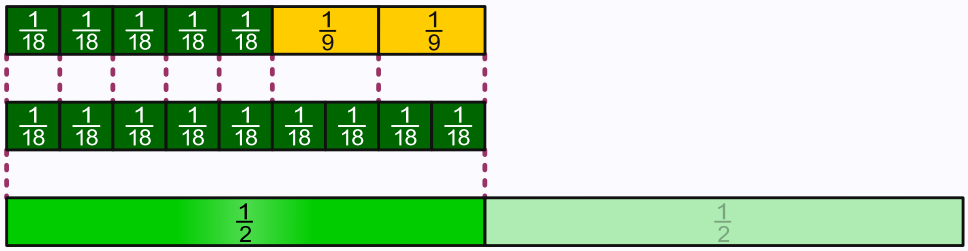
$$\frac{6}{21}+\frac{13}{26}=\frac{6\div 3}{21\div 3}+\frac{13\div 13}{26\div 13}=\frac{2}{7}+\frac{1}{2}=\frac{2\times 2}{7\times 2}+\frac{1\times 7}{2\times 7}=\frac{4}{14}+\frac{7}{14}=\frac{11}{14}$$



3. At the End: Simplify

After we add fractions, we write the result in its simplest form. That means: If we can simplify the result, we do it. For example:

$$\frac{5}{18}+\frac{2}{9}=\frac{5}{18}+\frac{2\times 2}{9\times 2}=\frac{5}{18}+\frac{4}{18}=\frac{9}{18}=\frac{9\div 9}{18\div 9}=\frac{1}{2}$$

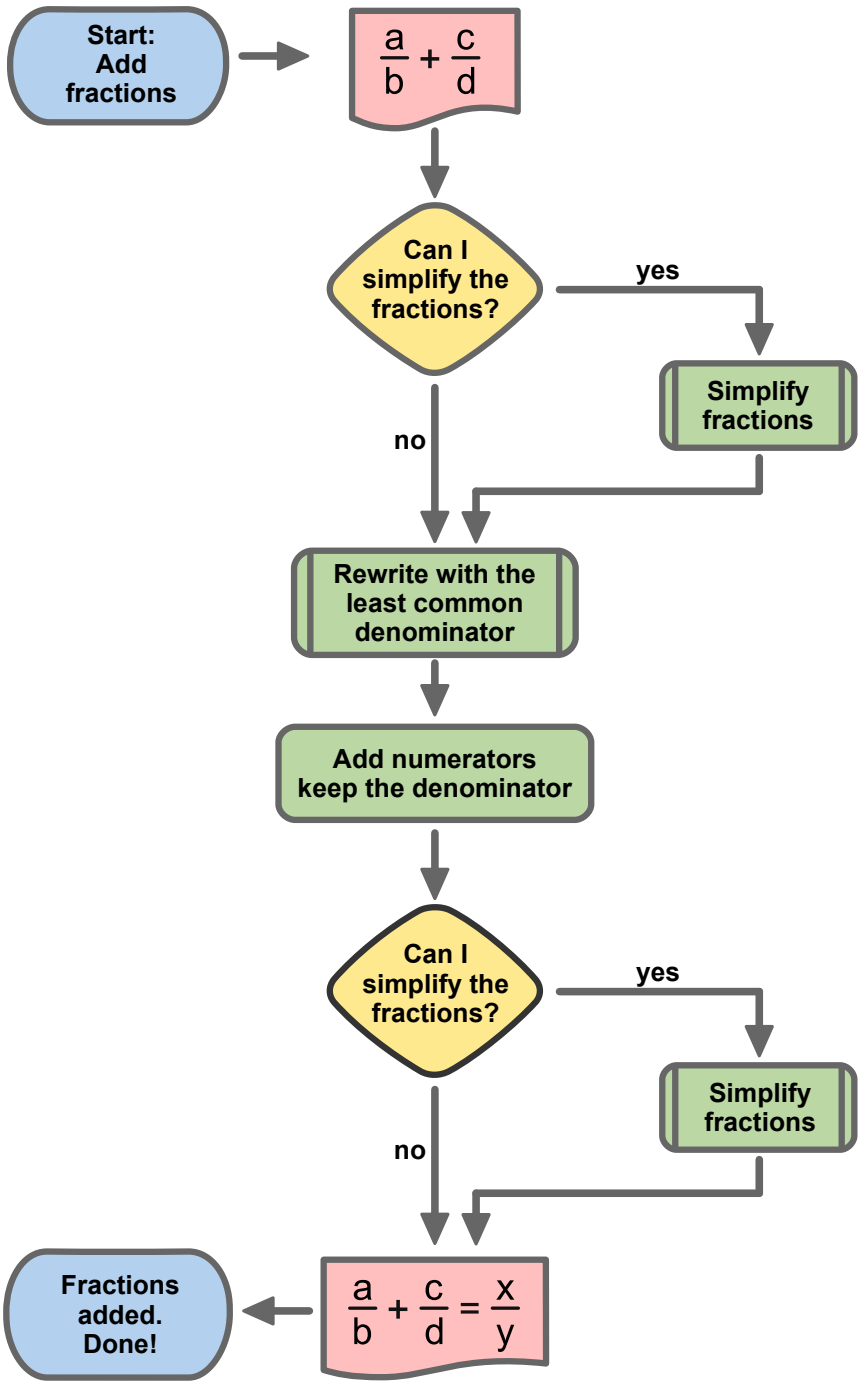


If the result is a whole number, we write that number instead of a fraction. For example:

$$\frac{3}{4}+\frac{5}{4}=\frac{3+5}{4}=\frac{8}{4}=\frac{8\div 4}{4\div 4}=\frac{2}{1}=2$$

7.2 Flowchart

The process of adding fractions with all necessary steps is shown in the following flowchart.



7.3 Examples

Let's look at some examples.

Example 1

1. Goal?

We want to add the two fractions $\frac{1}{8}$ and $\frac{3}{9}$.

2. Simplify?

$\frac{3}{9} = \frac{3 \div 3}{9 \div 3} = \frac{1}{3}$ From now on, we will add $\frac{1}{8}$ and $\frac{1}{3}$.

3. Least Common Denominator

$\frac{1}{8} = \frac{1 \times 3}{8 \times 3} = \frac{3}{24}$, $\frac{1}{3} = \frac{1 \times 8}{3 \times 8} = \frac{8}{24}$

4. Add

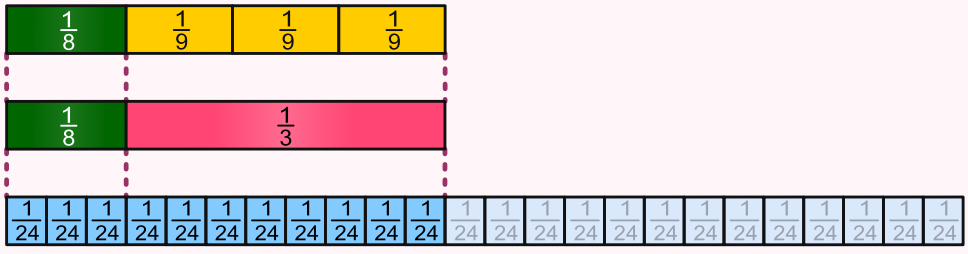
$\frac{3}{24} + \frac{8}{24} = \frac{11}{24}$

5. Simplify?

$\frac{11}{24}$ cannot be simplified.

6. Final Result

$$\frac{1}{8} + \frac{3}{9} = \frac{1}{8} + \frac{3 \div 3}{9 \div 3} = \frac{1}{8} + \frac{1}{3} = \frac{1 \times 3}{8 \times 3} + \frac{1 \times 8}{3 \times 8} = \frac{3}{24} + \frac{8}{24} = \frac{11}{24}$$



Example 2

1. Goal?

We want to add the two fractions $\frac{7}{9}$ and $\frac{1}{2}$.

2. Simplify?

$\frac{7}{9}$ and $\frac{1}{2}$ cannot be simplified.

3. Least Common Denominator

$$\frac{7}{9} = \frac{7 \times 2}{9 \times 2} = \frac{14}{18} \quad , \quad \frac{1}{2} = \frac{1 \times 9}{2 \times 9} = \frac{9}{18}$$

4. Add

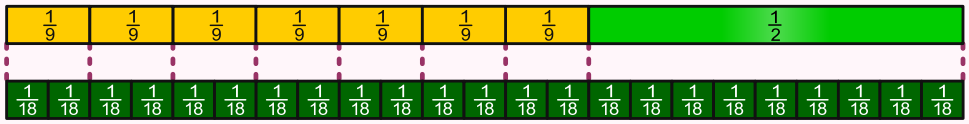
$$\frac{14}{18} + \frac{9}{18} = \frac{23}{18}$$

5. Simplify?

$\frac{23}{18}$ cannot be simplified.

6. Final Result

$$\frac{7}{9} + \frac{1}{2} = \frac{7 \times 2}{9 \times 2} + \frac{1 \times 9}{2 \times 9} = \frac{14}{18} + \frac{9}{18} = \frac{23}{18}$$



Example 3

1. Goal?

We want to add the two fractions $\frac{3}{18}$ and $\frac{6}{20}$.

2. Simplify?

$$\frac{3}{18} = \frac{3\div 3}{18\div 3} = \frac{1}{6} \quad \text{and} \quad \frac{6}{20} = \frac{6\div 2}{20\div 2} = \frac{3}{10}$$

3. Least Common Denominator

$$\frac{1}{6} = \frac{1\times 5}{6\times 5} = \frac{5}{30} \quad , \quad \frac{3}{10} = \frac{3\times 3}{10\times 3} = \frac{9}{30}$$

4. Add

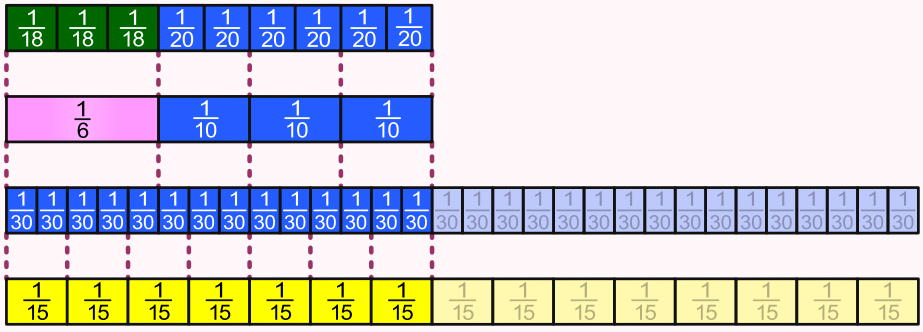
$$\frac{5}{30} + \frac{9}{30} = \frac{14}{30}$$

5. Simplify?

$$\frac{14}{30} = \frac{14\div 2}{30\div 2} = \frac{7}{15}$$

6. Final Result

$$\frac{3}{18}+\frac{6}{20} = \frac{3\div 3}{18\div 3}+\frac{6\div 2}{20\div 2} = \frac{1}{6}+\frac{3}{10} = \frac{1\times 5}{6\times 5}+\frac{3\times 3}{10\times 3} = \frac{5}{30}+\frac{9}{30} = \frac{14}{30} = \frac{14\div 2}{30\div 2} = \frac{7}{15}$$



7.4 Exercises

Exercise 1 - Fully Completed Worksheet

Exercise 1

1. Goal?

We want to add the fractions $\frac{21}{56}$ and $\frac{10}{16}$.

2. Simplify?

$$\frac{21}{56} = \frac{21 \div 7}{56 \div 7} = \frac{3}{8} \quad \text{and} \quad \frac{10}{16} = \frac{10 \div 2}{16 \div 2} = \frac{5}{8}$$

3. Least Common Denominator

$$\frac{3}{8} \quad , \quad \frac{5}{8}$$

4. Add

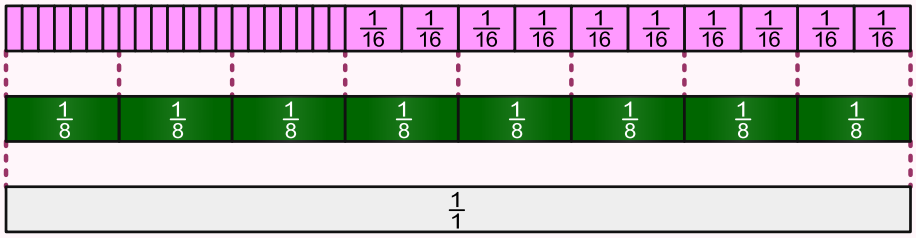
$$\frac{3}{8} + \frac{5}{8} = \frac{8}{8}$$

5. Simplify?

$$\frac{8}{8} = \frac{8 \div 8}{8 \div 8} = \frac{1}{1} = 1$$

6. Result

$$\frac{21}{56} + \frac{10}{16} = \frac{21 \div 7}{56 \div 7} + \frac{10 \div 2}{16 \div 2} = \frac{3}{8} + \frac{5}{8} = \frac{8}{8} = \frac{8 \div 8}{8 \div 8} = \frac{1}{1} = 1$$



Exercise 1 -Level 1

1. Goal?

$$\frac{21}{56} + \frac{10}{16}$$

2. Simpliy?

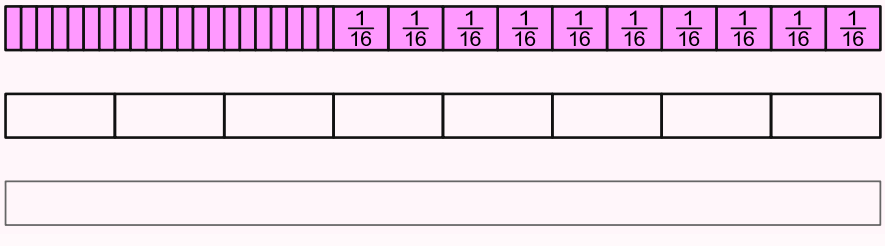
$\frac{21}{56} =$ $\frac{10}{16} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 1 -Level 2

1. Goal?

$$\frac{21}{56} + \frac{10}{16}$$

2. Simpliy?

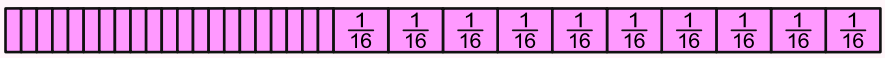
$\frac{21}{56} =$ $\frac{10}{16} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 1 -Level 3

1. Goal?

$$\frac{21}{56} + \frac{10}{16}$$

2. Simpliy?

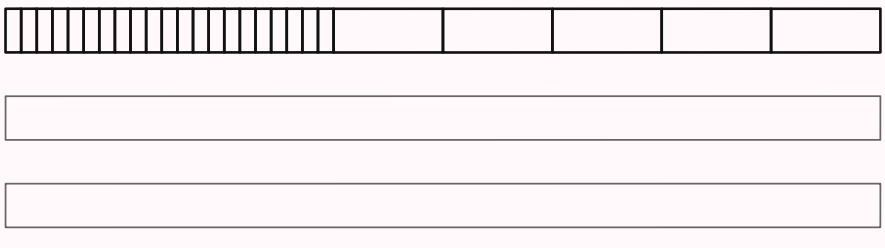
$\frac{21}{56} =$ $\frac{10}{16} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 2 - Fully Completed Worksheet

Exercise 2

1. Goal?

We want to add the fractions $\frac{6}{21}$ and $\frac{6}{28}$.

2. Simplify?

$$\frac{6}{21} = \frac{6 \div 3}{21 \div 3} = \frac{2}{7} \quad \text{and} \quad \frac{6}{28} = \frac{6 \div 2}{28 \div 2} = \frac{3}{14}$$

3. Least Common Denominator

$$\frac{2}{7} = \frac{2 \times 2}{7 \times 2} = \frac{4}{14} \quad , \quad \frac{3}{14}$$

4. Add

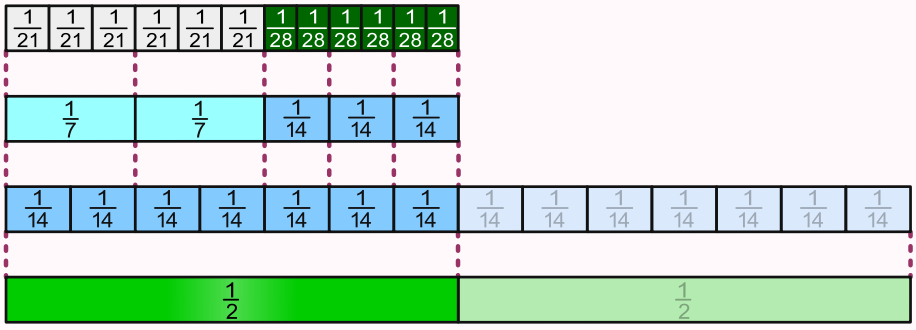
$$\frac{4}{14} + \frac{3}{14} = \frac{7}{14}$$

5. Simplify?

$$\frac{7}{14} = \frac{7 \div 7}{14 \div 7} = \frac{1}{2}$$

6. Result

$$\frac{6}{21}+\frac{6}{28} = \frac{6 \div 3}{21 \div 3}+\frac{6 \div 2}{28 \div 2} = \frac{2}{7}+\frac{3}{14} = \frac{2 \times 2}{7 \times 2}+\frac{3}{14} = \frac{4}{14}+\frac{3}{14} = \frac{7}{14} = \frac{7 \div 7}{14 \div 7} = \frac{1}{2}$$



Exercise 2 -Level 1

1. Goal?

$$\frac{6}{21} + \frac{6}{28}$$

2. Simpliy?

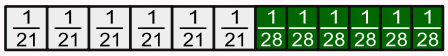
$\frac{6}{21} =$ $\frac{6}{28} =$

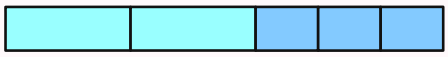
3. Least Common Denominator

4. Addieren

5. Simpliy?

6. Result





Exercise 2 -Level 2

1. Goal?

$$\frac{6}{21} + \frac{6}{28}$$

2. Simpliy?

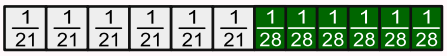
$\frac{6}{21} =$ $\frac{6}{28} =$

3. Least Common Denominator

4. Addieren

5. Simpliy?

6. Result



Exercise 2 -Level 3

1. Goal?

$$\frac{6}{21} + \frac{6}{28}$$

2. Simpliy?

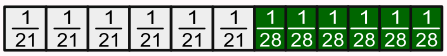
$\frac{6}{21} =$ $\frac{6}{28} =$

3. Least Common Denominator

4. Addieren

5. Simpliy?

6. Result



Exercise 3 - Fully Completed Worksheet

Exercise 3

1. Goal?

We want to add the fractions $\frac{12}{45}$ and $\frac{10}{12}$.

2. Simplify?

$$\frac{12}{45} = \frac{12 \div 3}{45 \div 3} = \frac{4}{15} \text{ and } \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

3. Least Common Denominator

$$\frac{4}{15} = \frac{4 \times 2}{15 \times 2} = \frac{8}{30} \quad , \quad \frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$$

4. Add

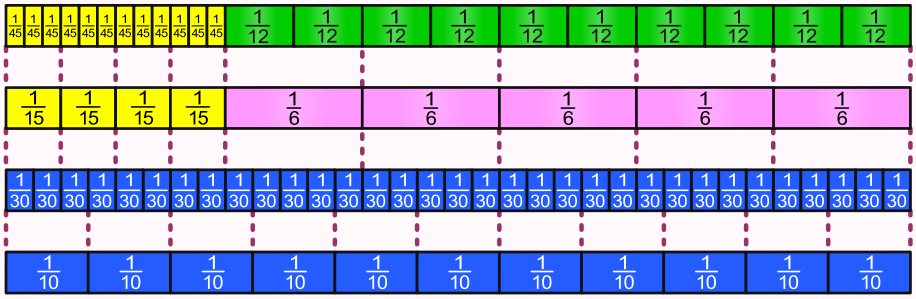
$$\frac{8}{30} + \frac{25}{30} = \frac{33}{30}$$

5. Simplify?

$$\frac{33}{30} = \frac{33 \div 3}{30 \div 3} = \frac{11}{10}$$

6. Result

$$\frac{12}{45}+\frac{10}{12} = \frac{12 \div 3}{45 \div 3}+\frac{10 \div 2}{12 \div 2} = \frac{4}{15}+\frac{5}{6} = \frac{4 \times 2}{15 \times 2}+\frac{5 \times 5}{6 \times 5} = \frac{8}{30}+\frac{25}{30} = \frac{33}{30} = \frac{33 \div 3}{30 \div 3} = \frac{11}{10}$$



Exercise 3 -Level 1

1. Goal?

$$\frac{12}{45} + \frac{10}{12}$$

2. Simpliy?

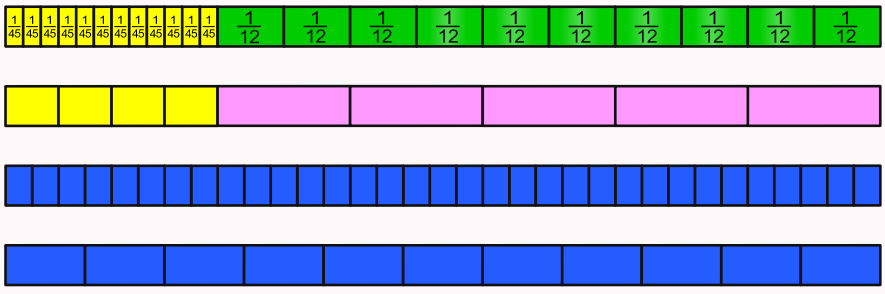
$\frac{12}{45} =$ $\frac{10}{12} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 3 -Level 2

1. Goal?

$$\frac{12}{45} + \frac{10}{12}$$

2. Simpliy?

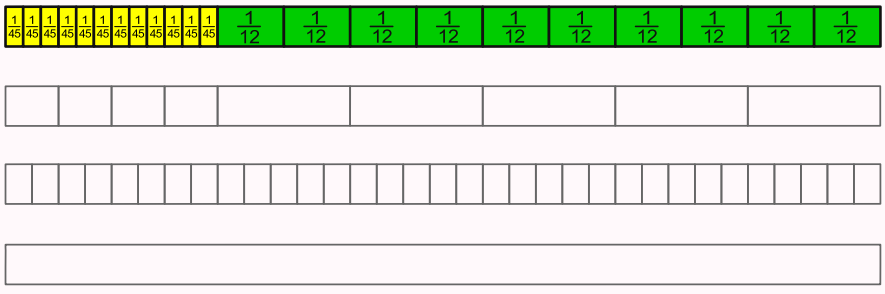
$\frac{12}{45} =$ $\frac{10}{12} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 3 -Level 3

1. Goal?

$$\frac{12}{45} + \frac{10}{12}$$

2. Simpliy?

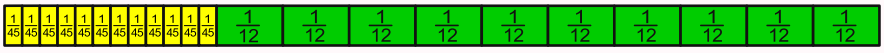
$\frac{12}{45} =$ $\frac{10}{12} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 4 - Fully Completed Worksheet

Exercise 4

1. Goal?

We want to add the fractions $\frac{6}{27}$ and $\frac{1}{36}$.

2. Simplify?

$$\frac{6}{27} = \frac{6 \div 3}{27 \div 3} = \frac{2}{9} \text{ and } \frac{1}{36}$$

3. Least Common Denominator

$$\frac{2}{9} = \frac{2 \times 4}{9 \times 4} = \frac{8}{36} \ , \ \frac{1}{36}$$

4. Add

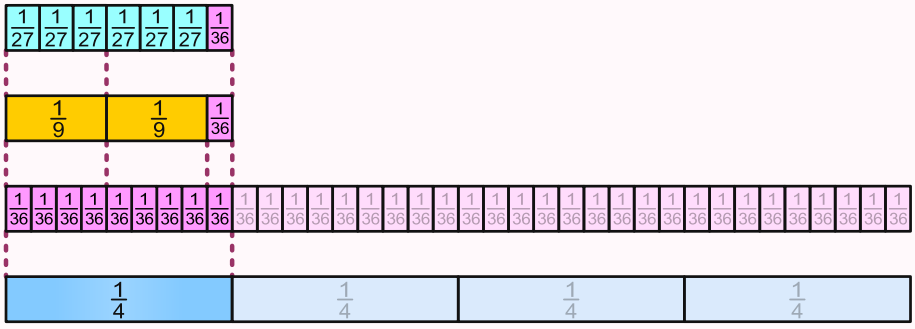
$$\frac{8}{36} + \frac{1}{36} = \frac{9}{36}$$

5. Simplify?

$$\frac{9}{36} = \frac{9 \div 9}{36 \div 9} = \frac{1}{4}$$

6. Result

$$\frac{6}{27}+\frac{1}{36} = \frac{6 \div 3}{27 \div 3}+\frac{1}{36} = \frac{2}{9}+\frac{1}{36} = \frac{2 \times 4}{9 \times 4}+\frac{1}{36} = \frac{8}{36}+\frac{1}{36} = \frac{9}{36} = \frac{9 \div 9}{36 \div 9} = \frac{1}{4}$$



Exercise 4 -Level 1

1. Goal?

$$\frac{6}{27} + \frac{1}{36}$$

2. Simpliy?

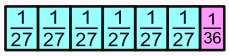
$\frac{6}{27} =$ $\frac{1}{36} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 4 -Level 2

1. Goal?

$$\frac{6}{27} + \frac{1}{36}$$

2. Simpliy?

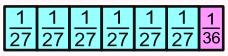
$\frac{6}{27} =$ $\frac{1}{36} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 4 -Level 3

1. Goal?

$$\frac{6}{27} + \frac{1}{36}$$

2. Simpliy?

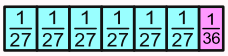
$\frac{6}{27} =$ $\frac{1}{36} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 5 - Fully Completed Worksheet

Exercise 5

1. Goal?

We want to add the fractions $\frac{6}{105}$ and $\frac{27}{42}$.

2. Simplify?

$$\frac{6}{105} = \frac{6 \div 3}{105 \div 3} = \frac{2}{35} \quad \text{and} \quad \frac{27}{42} = \frac{27 \div 3}{42 \div 3} = \frac{9}{14}$$

3. Least Common Denominator

$$\frac{2}{35} = \frac{2 \times 2}{35 \times 2} = \frac{4}{70} \quad , \quad \frac{9}{14} = \frac{9 \times 5}{14 \times 5} = \frac{45}{70}$$

4. Add

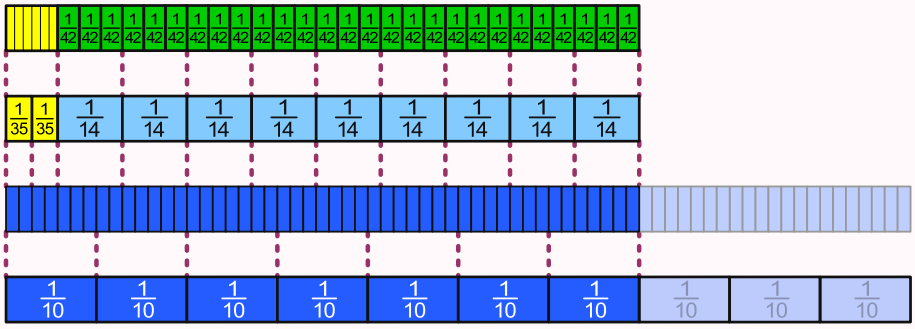
$$\frac{4}{70} + \frac{45}{70} = \frac{49}{70}$$

5. Simplify?

$$\frac{49}{70} = \frac{49 \div 7}{70 \div 7} = \frac{7}{10}$$

6. Result

$$\frac{6}{105}+\frac{27}{42} = \frac{6 \div 3}{105 \div 3}+\frac{27 \div 3}{42 \div 3} = \frac{2}{35}+\frac{9}{14} = \frac{2 \times 2}{35 \times 2}+\frac{9 \times 5}{14 \times 5} = \frac{4}{70}+\frac{45}{70} = \frac{49}{70} = \frac{49 \div 7}{70 \div 7} = \frac{7}{10}$$



Exercise 5 -Level 1

1. Goal?

$$\frac{6}{105} + \frac{27}{42}$$

2. Simpliy?

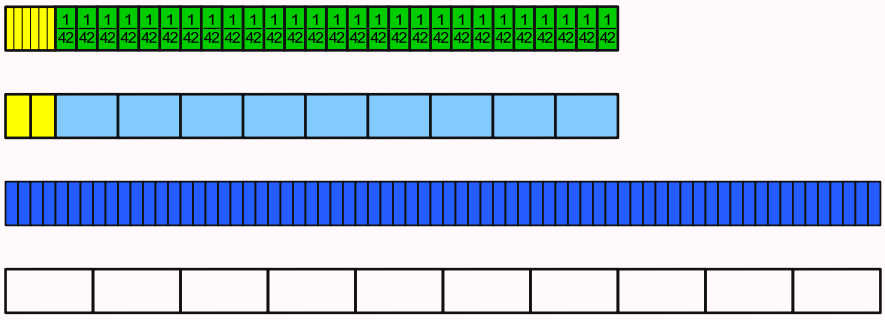
$\frac{6}{105} =$ $\frac{27}{42} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 5 -Level 2

1. Goal?

$$\frac{6}{105} + \frac{27}{42}$$

2. Simpliy?

$\frac{6}{105} =$ $\frac{27}{42} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result

Exercise 5 -Level 3

1. Goal?

$$\frac{6}{105} + \frac{27}{42}$$

2. Simpliy?

$\frac{6}{105} =$ $\frac{27}{42} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result

Exercise 6 - Fully Completed Worksheet

Exercise 6

1. Goal?

We want to add the fractions $\frac{8}{42}$ and $\frac{26}{39}$.

2. Simplify?

$$\frac{8}{42} = \frac{8 \div 2}{42 \div 2} = \frac{4}{21} \quad \text{and} \quad \frac{26}{39} = \frac{26 \div 13}{39 \div 13} = \frac{2}{3}$$

3. Least Common Denominator

$$\frac{4}{21} \ , \ \frac{2}{3} = \frac{2 \times 7}{3 \times 7} = \frac{14}{21}$$

4. Add

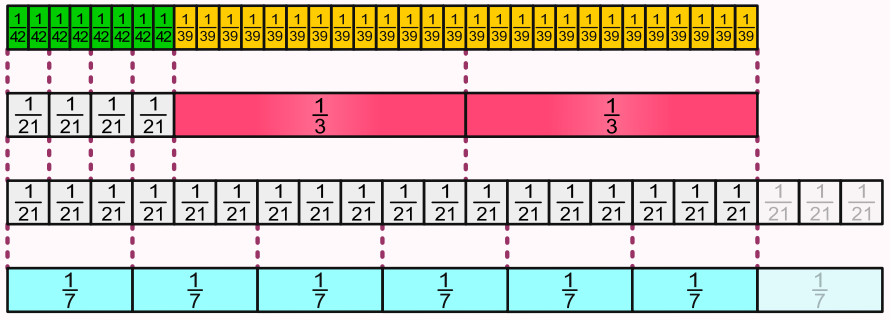
$$\frac{4}{21} + \frac{14}{21} = \frac{18}{21}$$

5. Simplify?

$$\frac{18}{21} = \frac{18 \div 3}{21 \div 3} = \frac{6}{7}$$

6. Final result

$$\frac{8}{42}+\frac{26}{39} = \frac{8 \div 2}{42 \div 2}+\frac{26 \div 13}{39 \div 13} = \frac{4}{21}+\frac{2}{3} = \frac{4}{21}+\frac{2 \times 7}{3 \times 7} = \frac{4}{21}+\frac{14}{21} = \frac{18}{21} = \frac{18 \div 3}{21 \div 3} = \frac{6}{7}$$



Exercise 6 -Level 1

1. Goal?

$$\frac{8}{42} + \frac{26}{39}$$

2. Simpliy?

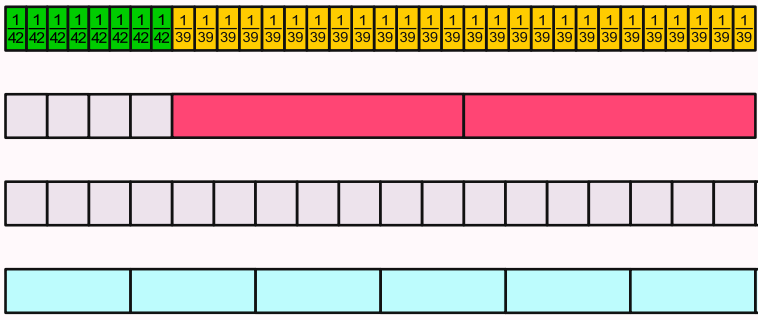
$\frac{8}{42} =$ $\frac{26}{39} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 6 -Level 2

1. Goal?

$$\frac{8}{42} + \frac{26}{39}$$

2. Simpliy?

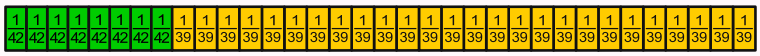
$\frac{8}{42} =$ $\frac{26}{39} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



Exercise 6 -Level 3

1. Goal?

$$\frac{8}{42} + \frac{26}{39}$$

2. Simpliy?

$\frac{8}{42} =$

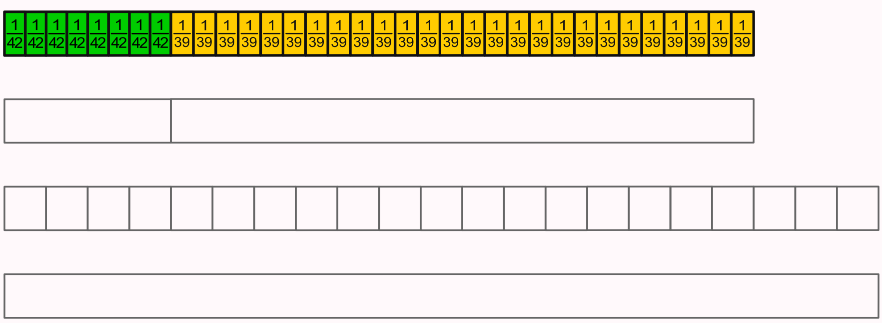
$\frac{26}{39} =$

3. Least Common Denominator

4. Add

5. Simpliy?

6. Result



8 Subtracting Fractions

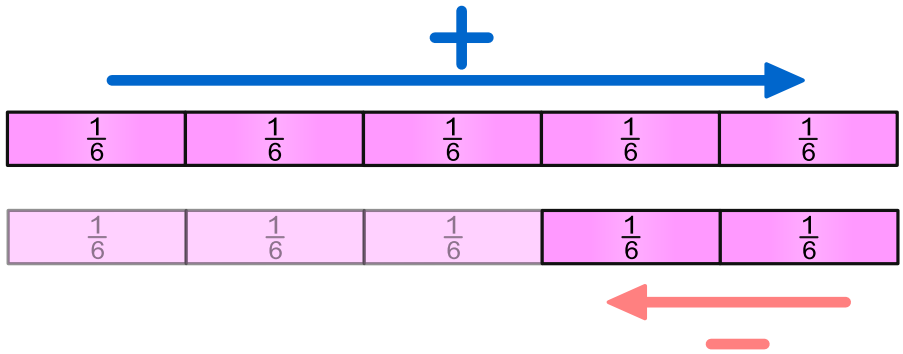
8.1 Subtracting Like Fractions

To visualize the addition of fractions using fraction strips, we simply placed two fraction strips next to each other. To visualize the subtraction of fractions using fraction strips, we can think of the fractional parts as steps:

The expression $\frac{5}{6} - \frac{2}{6}$ then becomes:

First, we take 5 sixth-steps to the right and then 2 sixth-steps to the left. We end up at 3 sixth-steps, and that is the result of the subtraction.

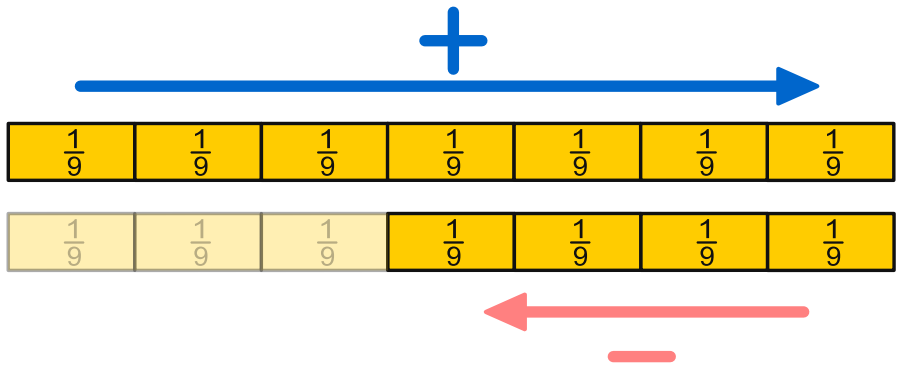
So: $\frac{5}{6} - \frac{2}{6} = \frac{3}{6}$.



When we added like fractions, we added the numerators and kept the common denominator. When we subtract like fractions, we proceed in a similar way: we subtract the numerators and keep the common denominator. The calculation looks like this:

$$\frac{5}{6} - \frac{2}{6} = \frac{5-2}{6} = \frac{3}{6}$$

Let's take a look at another example of subtraction:



Here we first take 7 steps of ninths to the right and then 4 steps of ninths to the left. We end up at 3 ninths.
We can also express this situation using numbers:

$$\frac{7}{9} - \frac{4}{9} = \frac{7-4}{9} = \frac{3}{9}$$

When subtracting fractions, we can proceed very similarly to how we add fractions.

> Two like fractions are subtracted
> by subtracting the numerators.
> The denominator stays the same.

We write this as a formula:

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

As usual, we call the result of a subtraction the *difference*.

We always write the difference as a simplified fraction. For example:

$$\frac{11}{8} - \frac{5}{8} = \frac{11-5}{8} = \frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

If the difference is a whole number, then we write the result as that whole number. For example:

$$\frac{15}{4} - \frac{3}{4} = \frac{15-3}{4} = \frac{12}{4} = \frac{12 \div 4}{4 \div 4} = \frac{3}{1} = 3$$

8.2 Subtracting Unlike Fractions

When we want to subtract unlike fractions, we first rewrite the fractions with a common denominator, and then subtract the like fractions. For example:

$$\frac{5}{8} - \frac{3}{20} = \frac{5 \times 5}{8 \times 5} - \frac{3 \times 2}{20 \times 2} = \frac{25}{40} - \frac{6}{40} = \frac{25-6}{40} = \frac{19}{40}$$

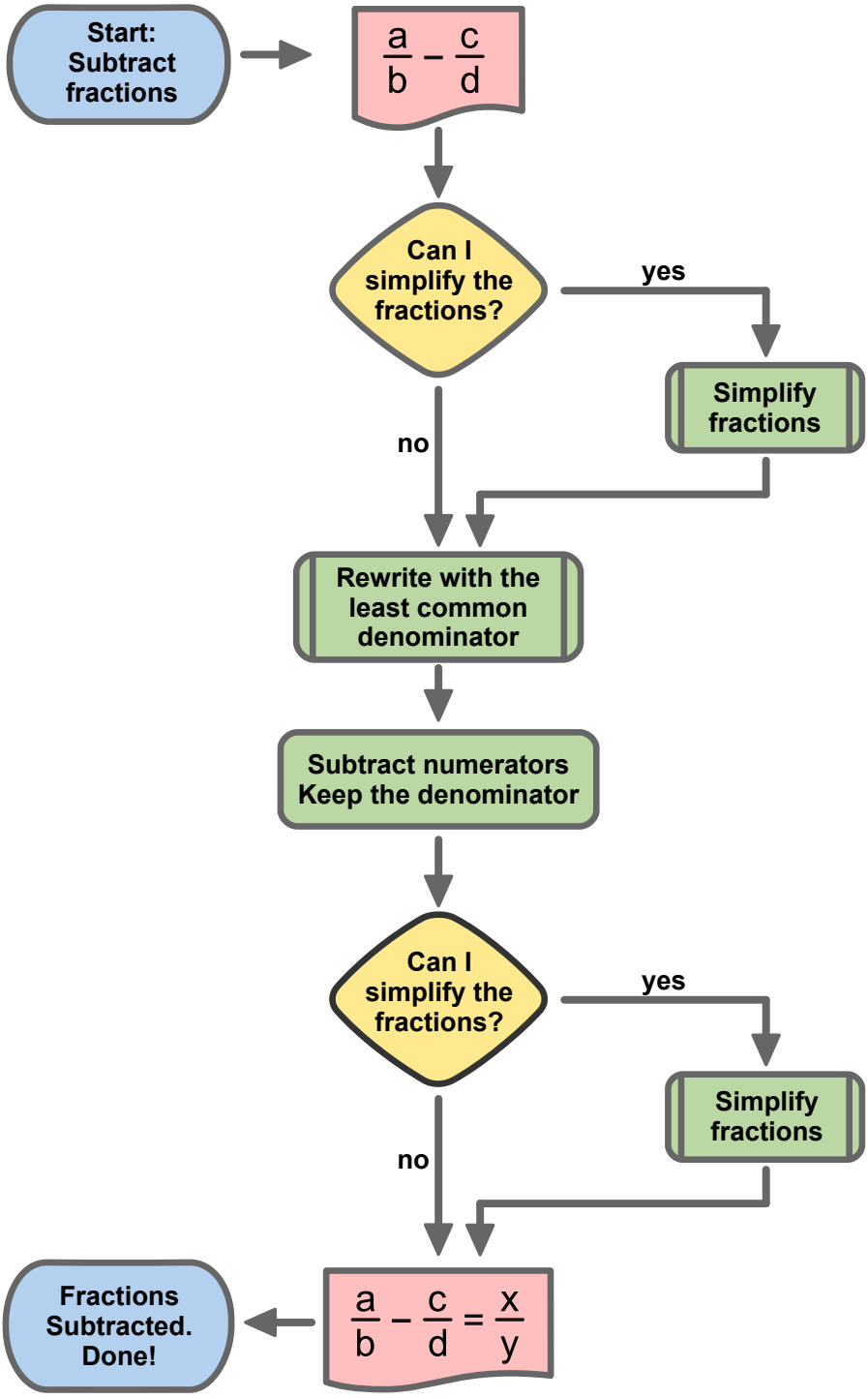
and

$$\frac{9}{10} - \frac{6}{7} = \frac{9 \times 7}{10 \times 7} - \frac{6 \times 10}{7 \times 10} = \frac{63}{70} - \frac{60}{70} = \frac{63-60}{70} = \frac{3}{70}$$

.

8.3 Flowchart

The flowchart shows the procedure we use to subtract fractions.



8.4 Examples

Let's look at a few examples.

Example 1

1. Goal?

We want to subtract $\frac{1}{3}$ from $\frac{3}{4}$, which means we want to calculate $\frac{3}{4} - \frac{1}{3}$.

2. Simplify?

Neither $\frac{3}{4}$ nor $\frac{1}{3}$ can be simplified.

3. Least Common Denominator

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12} \quad , \quad \frac{1}{3} = \frac{1 \times 4}{3 \times 4} = \frac{4}{12}$$

4. Subtract

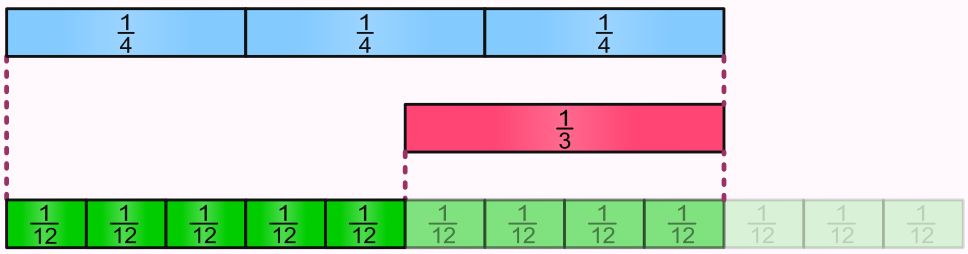
$$\frac{9}{12} - \frac{4}{12} = \frac{5}{12}$$

5. Simplify?

$\frac{5}{12}$ cannot be simplified.

6. Final Result

$$\frac{3}{4} - \frac{1}{3} = \frac{3 \times 3}{4 \times 3} - \frac{1 \times 4}{3 \times 4} = \frac{9}{12} - \frac{4}{12} = \frac{5}{12}$$



Example 2

1. Goal?

We want to subtract $\frac{4}{6}$ from $\frac{14}{15}$, which means we want to calculate $\frac{14}{15} - \frac{4}{6}$.

2. Simplify?

$\frac{14}{15}$ cannot be simplified; $\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$

3. Least Common Denominator

$\frac{14}{15}$, $\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$

4. Subtract

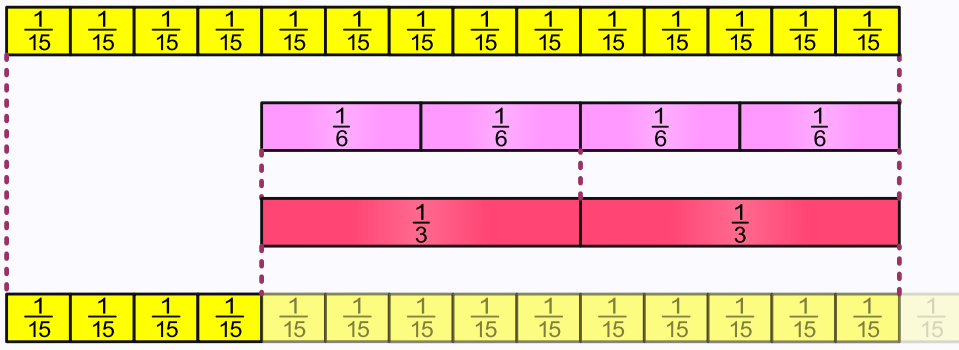
$\frac{14}{15} - \frac{10}{15} = \frac{4}{15}$

5. Simplify?

$\frac{4}{15}$ cannot be simplified.

6. Final Result

$$\frac{14}{15} - \frac{4}{6} = \frac{14}{15} - \frac{4 \div 2}{6 \div 2} = \frac{14}{15} - \frac{2}{3} = \frac{14}{15} - \frac{2 \times 5}{3 \times 5} = \frac{14}{15} - \frac{10}{15} = \frac{4}{15}$$



Example 3

1. Goal?

We want to subtract $\frac{13}{26}$ from $\frac{18}{21}$, which means we want to calculate $\frac{18}{21} - \frac{13}{26}$.

2. Simplify?

$$\frac{18}{21} = \frac{18 \div 3}{21 \div 3} = \frac{6}{7} \text{ and } \frac{13}{26} = \frac{13 \div 13}{26 \div 13} = \frac{1}{2}$$

3. Least Common Denominator

$$\frac{6}{7} = \frac{6 \times 2}{7 \times 2} = \frac{12}{14} \quad , \quad \frac{1}{2} = \frac{1 \times 7}{2 \times 7} = \frac{7}{14}$$

4. Subtract

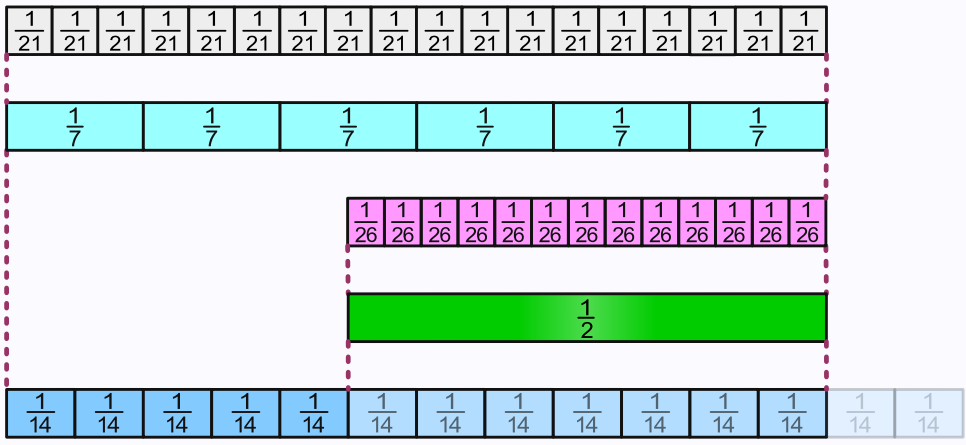
$$\frac{12}{14} - \frac{7}{14} = \frac{5}{14}$$

5. Simplify?

$\frac{5}{14}$ cannot be simplified.

6. Result

$$\frac{18}{21} - \frac{13}{26} = \frac{18 \div 3}{21 \div 3} - \frac{13 \div 13}{26 \div 13} = \frac{6}{7} - \frac{1}{2} = \frac{6 \times 2}{7 \times 2} - \frac{1 \times 7}{2 \times 7} = \frac{12}{14} - \frac{7}{14} = \frac{5}{14}$$



Example 4

1. Goal?

We want to subtract $\frac{20}{75}$ from $\frac{22}{24}$, which means we want to calculate $\frac{22}{24} - \frac{20}{75}$.

2. Simplify?

$$\frac{22}{24} = \frac{22 \div 2}{24 \div 2} = \frac{11}{12} \quad \text{and} \quad \frac{20}{75} = \frac{20 \div 5}{75 \div 5} = \frac{4}{15}$$

3. Least Common Denominator

$$\frac{11}{12} = \frac{11 \times 5}{12 \times 5} = \frac{55}{60} \quad , \quad \frac{4}{15} = \frac{4 \times 4}{15 \times 4} = \frac{16}{60}$$

4. Subtract

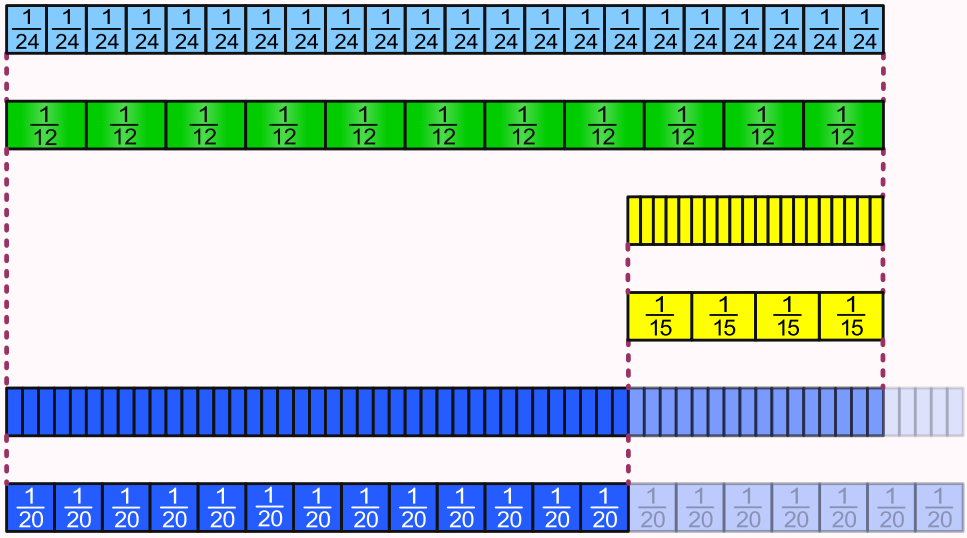
$$\frac{55}{60} - \frac{16}{60} = \frac{39}{60}$$

5. Simplify?

$$\frac{39}{60} = \frac{39 \div 3}{60 \div 3} = \frac{13}{20}$$

6. Final Result

$$\frac{22}{24}-\frac{20}{75}=\frac{22\div 2}{24\div 2}-\frac{20\div 5}{75\div 5}=\frac{11}{12}-\frac{4}{15}=\frac{11\times 5}{12\times 5}-\frac{4\times 4}{15\times 4}=\frac{55}{60}-\frac{16}{60}=\frac{39}{60}=\frac{39\div 3}{60\div 3}=\frac{13}{20}$$



Example 5

1. Goal?

We want to subtract $\frac{10}{12}$ from $\frac{42}{45}$, so we calculate $\frac{42}{45} - \frac{10}{12}$.

2. Simplify?

$$\frac{42}{45} = \frac{42 \div 3}{45 \div 3} = \frac{14}{15} \quad \text{and} \quad \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

3. Least Common Denominator

$$\frac{14}{15} = \frac{14 \times 2}{15 \times 2} = \frac{28}{30} \quad , \quad \frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$$

4. Subtract

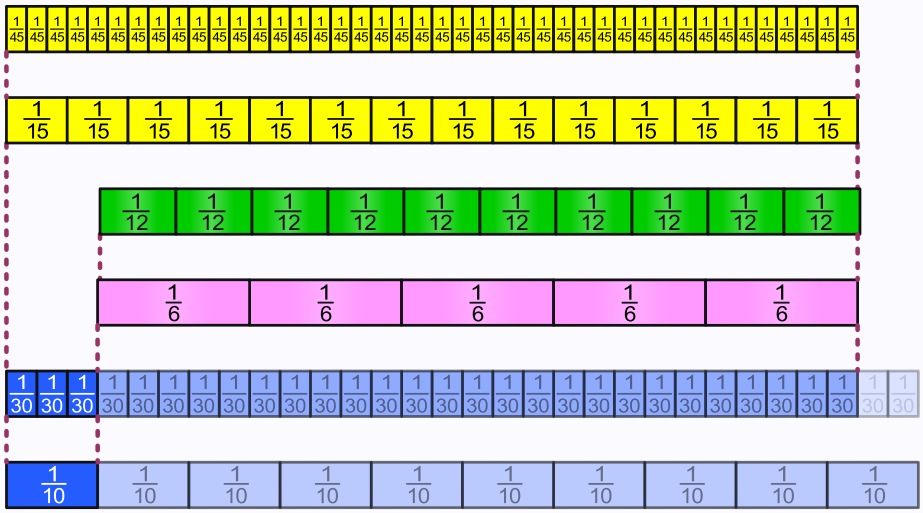
$$\frac{28}{30} - \frac{25}{30} = \frac{3}{30}$$

5. Simplify?

$$\frac{3}{30} = \frac{3 \div 3}{30 \div 3} = \frac{1}{10}$$

6. Final Result

$$\frac{42}{45}-\frac{10}{12}=\frac{42\div 3}{45\div 3}-\frac{10\div 2}{12\div 2}=\frac{14}{15}-\frac{5}{6}=\frac{14\times 2}{15\times 2}-\frac{5\times 5}{6\times 5}=\frac{28}{30}-\frac{25}{30}=\frac{3}{30}=\frac{3\div 3}{30\div 3}=\frac{1}{10}$$



8.5 Exercises

Exercise 1

1. Goal?

We want to subtract $\frac{1}{6}$ from $\frac{2}{3}$, so we calculate $\frac{2}{3} - \frac{1}{6}$.

2. Simplify?

Neither $\frac{2}{3}$ nor $\frac{1}{6}$ can be simplified.

3. Least Common Denominator

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} \quad , \quad \frac{1}{6}$$

4. Subtract

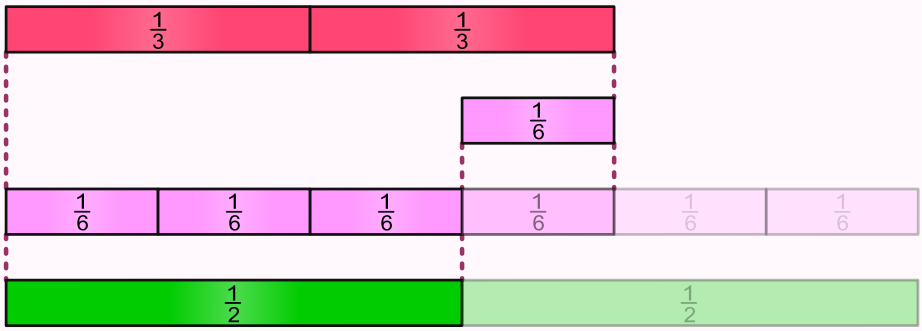
$$\frac{4}{6} - \frac{1}{6} = \frac{3}{6}$$

5. Simplify?

$$\frac{3}{6} = \frac{3 \div 3}{6 \div 3} = \frac{1}{2}$$

6. Final Result

$$\frac{2}{3} - \frac{1}{6} = \frac{2 \times 2}{3 \times 2} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{3 \div 3}{6 \div 3} = \frac{1}{2}$$



Exercise 2

1. Goal

We want to subtract $\frac{10}{34}$ from $\frac{15}{17}$, so we calculate $\frac{15}{17} - \frac{10}{34}$.

2. Simplify?

$\frac{15}{17}$ cannot be simplified; $\frac{10}{34} = \frac{10 \div 2}{34 \div 2} = \frac{5}{17}$

3. Least common denominator

$\frac{15}{17}$, $\frac{5}{17}$

4. Subtract

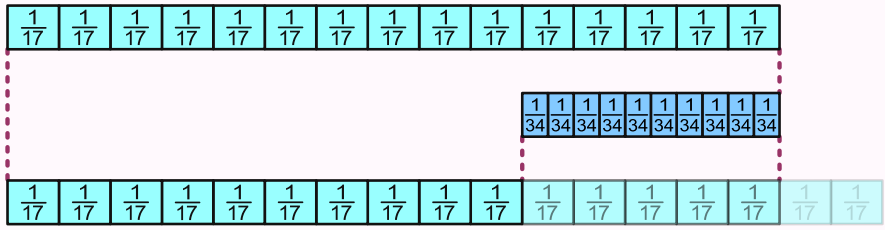
$$\frac{15}{17} - \frac{5}{17} = \frac{10}{17}$$

5. Simplify?

$\frac{10}{17}$ cannot be simplified.

6. Final Result

$$\frac{15}{17} - \frac{10}{34} = \frac{15}{17} - \frac{10 \div 2}{34 \div 2} = \frac{15}{17} - \frac{5}{17} = \frac{10}{17}$$



Exercise 3

1. Goal

We want to subtract $\frac{11}{20}$ from $\frac{22}{24}$, so we calculate $\frac{22}{24} - \frac{11}{20}$.

2. Simplify?

$$\frac{22}{24} = \frac{22 \div 2}{24 \div 2} = \frac{11}{12} \quad ; \quad \frac{11}{20}$$

3. Least common denominator

$$\frac{11}{12} = \frac{11 \times 5}{12 \times 5} = \frac{55}{60} \quad , \quad \frac{11}{20} = \frac{11 \times 3}{20 \times 3} = \frac{33}{60}$$

4. Subtract

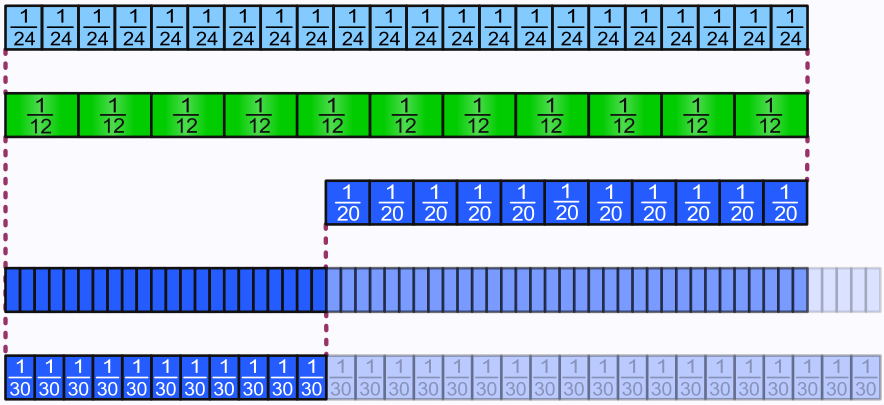
$$\frac{55}{60} - \frac{33}{60} = \frac{22}{60}$$

5. Simplify?

$$\frac{22}{60} = \frac{22 \div 2}{60 \div 2} = \frac{11}{30}$$

6. Final Result

$$\frac{22}{24} - \frac{11}{20} = \frac{22 \div 2}{24 \div 2} - \frac{11}{20} = \frac{11}{12} - \frac{11}{20} = \frac{11 \times 5}{12 \times 5} - \frac{11 \times 3}{20 \times 3} = \frac{55}{60} - \frac{33}{60} = \frac{22}{60} = \frac{22 \div 2}{60 \div 2} = \frac{11}{30}$$



Exercise 4

1. Goal

We want to subtract $\frac{20}{56}$ from $\frac{25}{30}$, so we calculate $\frac{25}{30} - \frac{20}{56}$.

2. Simplify?

$$\frac{25}{30} = \frac{25 \div 5}{30 \div 5} = \frac{5}{6} \quad \text{and} \quad \frac{20}{56} = \frac{20 \div 4}{56 \div 4} = \frac{5}{14}$$

3. Least common denominator

$$\frac{5}{6} = \frac{5 \times 7}{6 \times 7} = \frac{35}{42} \quad , \quad \frac{5}{14} = \frac{5 \times 3}{14 \times 3} = \frac{15}{42}$$

4. Subtract

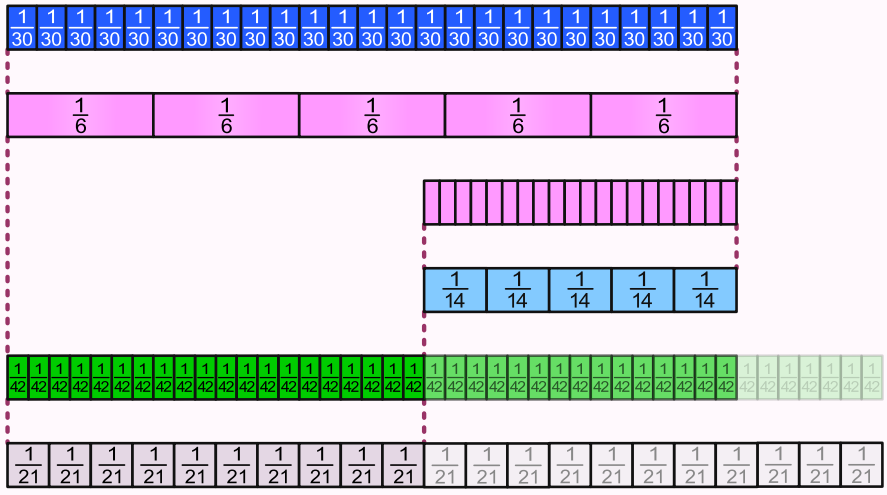
$$\frac{35}{42} - \frac{15}{42} = \frac{20}{42}$$

5. Simplify?

$$\frac{20}{42} = \frac{20 \div 2}{42 \div 2} = \frac{10}{21}$$

6. Final Result

$$\frac{25}{30} - \frac{20}{56} = \frac{25 \div 5}{30 \div 5} - \frac{20 \div 4}{56 \div 4} = \frac{5}{6} - \frac{5}{14} = \frac{5 \times 7}{6 \times 7} - \frac{5 \times 3}{14 \times 3} = \frac{35}{42} - \frac{15}{42} = \frac{20}{42} = \frac{20 \div 2}{42 \div 2} = \frac{10}{21}$$



Exercise 5

1. Goal

We want to subtract $\frac{2}{15}$ from $\frac{15}{50}$, so we calculate $\frac{15}{50} - \frac{2}{15}$.

2. Simplify?

$$\frac{15}{50} = \frac{15 \div 5}{50 \div 5} = \frac{3}{10} \; ; \; \frac{2}{15}$$

3. Least Common Denominator

$$\frac{3}{10} = \frac{3 \times 3}{10 \times 3} = \frac{9}{30} \quad , \quad \frac{2}{15} = \frac{2 \times 2}{15 \times 2} = \frac{4}{30}$$

4. Subtract

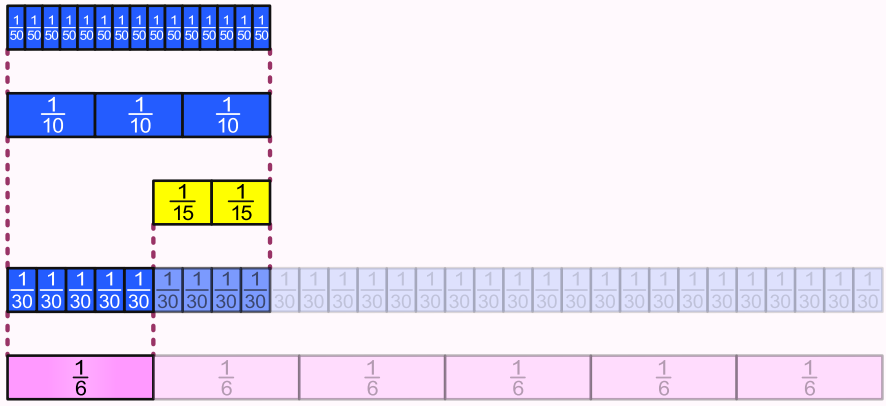
$$\frac{9}{30} - \frac{4}{30} = \frac{5}{30}$$

5. Simplify?

$$\frac{5}{30} = \frac{5 \div 5}{30 \div 5} = \frac{1}{6}$$

6. Final Result

$$\frac{15}{50} - \frac{2}{15} = \frac{15 \div 5}{50 \div 5} - \frac{2}{15} = \frac{3}{10} - \frac{2}{15} = \frac{3 \times 3}{10 \times 3} - \frac{2 \times 2}{15 \times 2} = \frac{9}{30} - \frac{4}{30} = \frac{5}{30} = \frac{5 \div 5}{30 \div 5} = \frac{1}{6}$$



Exercise 6

1. Goal

We want to subtract $\frac{39}{63}$ from $\frac{65}{60}$, so we calculate $\frac{65}{60} - \frac{39}{63}$.

2. Simplify?

$$\frac{65}{60} = \frac{65 \div 5}{60 \div 5} = \frac{13}{12} \; ; \; \frac{39}{63} = \frac{39 \div 3}{63 \div 3} = \frac{13}{21}$$

3. Least Common Denominator

$$\frac{13}{12} = \frac{13 \times 7}{12 \times 7} = \frac{91}{84} \quad , \quad \frac{13}{21} = \frac{13 \times 4}{21 \times 4} = \frac{52}{84}$$

4. Subtract

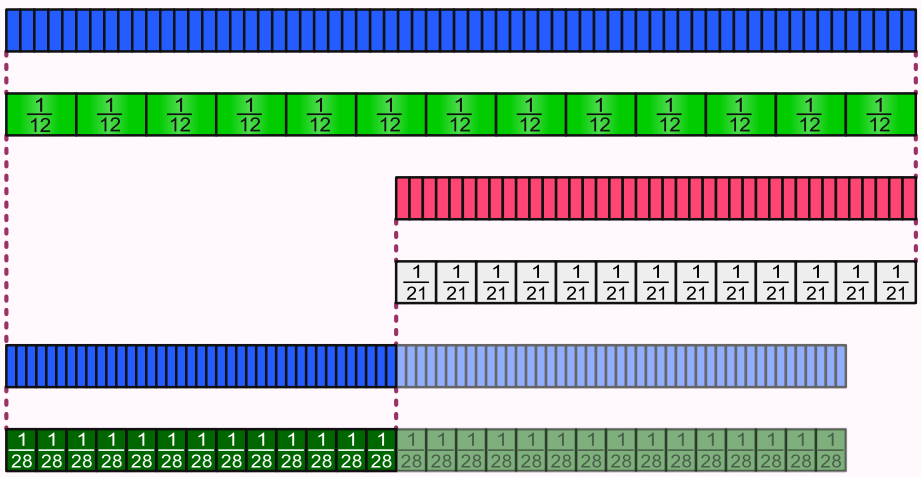
$$\frac{91}{84} - \frac{52}{84} = \frac{39}{84}$$

5. Simplify?

$$\frac{39}{84} = \frac{39 \div 3}{84 \div 3} = \frac{13}{28}$$

6. Final Result

$$\frac{65}{60} - \frac{39}{63} = \frac{65 \div 5}{60 \div 5} - \frac{39 \div 3}{63 \div 3} = \frac{13}{12} - \frac{13}{21} = \frac{13 \times 7}{12 \times 7} - \frac{13 \times 4}{21 \times 4} = \frac{91}{84} - \frac{52}{84} = \frac{39}{84} = \frac{39 \div 3}{84 \div 3} = \frac{13}{28}$$



Exercise 7

1. Goal?

We want to subtract $\frac{3}{7}$ from $\frac{3}{4}$, so we calculate $\frac{3}{4} - \frac{3}{7}$.

2. Simplify?

Neither $\frac{3}{4}$ nor $\frac{3}{7}$ can be simplified.

3. Least common denominator

$$\frac{3}{4} = \frac{3 \times 7}{4 \times 7} = \frac{21}{28} \quad , \quad \frac{3}{7} = \frac{3 \times 4}{7 \times 4} = \frac{12}{28}$$

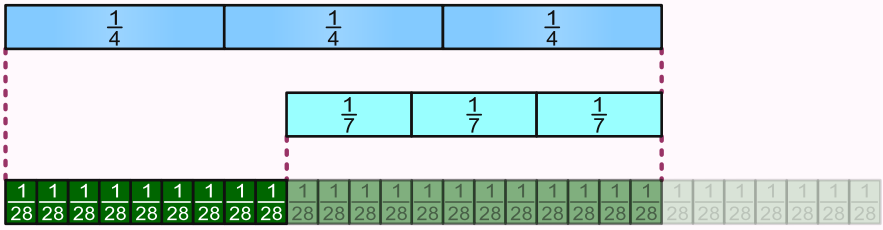
4. Subtract

$$\frac{21}{28} - \frac{12}{28} = \frac{9}{28}$$

5. Simplify? $\frac{9}{28}$ cannot be simplified.

6. Final Result

$$\frac{3}{4} - \frac{3}{7} = \frac{3 \times 7}{4 \times 7} - \frac{3 \times 4}{7 \times 4} = \frac{21}{28} - \frac{12}{28} = \frac{9}{28}$$



9 Multiplying Fractions

9.1 Definition

In mathematics, the following rule has become standard:
Fractions are multiplied by multiplying the numerators and the denominators.
This rule can be summarized briefly as:

> Numerator times numerator,
> denominator times denominator.

Using variables, we can write this very concisely as:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Here, a, b, c, and d stand for natural numbers, with b and d assumed to be nonzero.

9.2 Examples

$\frac{1}{2} \times \frac{1}{3} = \frac{1 \times 1}{2 \times 3} = \frac{1}{6}$

The numbers being multiplied are called **factors**. In this case, the factors $\frac{1}{2}$ and $\frac{1}{3}$ are being multiplied.

$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$

The result of a multiplication is called the **product**. Here, the product is the fraction $\frac{8}{15}$.

$\frac{4}{1} \times \frac{6}{1} = \frac{4 \times 6}{1 \times 1} = \frac{24}{1} = 24$

When we apply fraction multiplication to natural numbers, we get the results we are familiar with from whole-number multiplication, because $\frac{4}{1}$ equals 4, $\frac{6}{1}$ equals 6, $\frac{24}{1}$ equals 24, and 4×6 equals 24.

To avoid working with unnecessarily large numbers, we simplify fractions before multiplying them.

$$\frac{15}{9} \times \frac{14}{21} = \frac{15 \div 3}{9 \div 3} \times \frac{14 \div 7}{21 \div 7} = \frac{5}{3} \times \frac{2}{3} = \frac{5 \times 2}{3 \times 3} = \frac{10}{9}$$

If there's another opportunity to simplify, we take it.

$$\frac{6}{3} \times \frac{4}{8} = \frac{6 \div 3}{3 \div 3} \times \frac{12 \div 4}{8 \div 4} = \frac{2}{1} \times \frac{3}{2} = \frac{2 \times 3}{1 \times 2} = \frac{\cancel{2} \times 3}{1 \times \cancel{2}} = \frac{3}{1} = 3$$

Sometimes we write a number as a product of prime numbers to make simplification easier.

$$\frac{14}{21} \times \frac{25}{20} = \frac{14 \div 7}{21 \div 7} \times \frac{25 \div 5}{20 \div 5} = \frac{2}{3} \times \frac{5}{4} = \frac{2 \times 5}{3 \times 4} = \frac{2 \times 5}{3 \times 2 \times 2} = \frac{\cancel{2} \times 5}{3 \times \cancel{2} \times 2} = \frac{5}{3 \times 2} = \frac{5}{6}$$

Especially when working with "larger" numbers, it can be helpful to write both numerators and denominators as products of prime numbers right from the start.

$$\frac{14}{21} \times \frac{25}{20} = \frac{2 \times 7}{3 \times 7} \times \frac{5 \times 5}{2 \times 2 \times 5} = \frac{\cancel{2} \times \cancel{7}}{3 \times \cancel{7}} \times \frac{5 \times \cancel{5}}{\cancel{2} \times 2 \times \cancel{5}} = \frac{5}{3 \times 2} = \frac{5}{6}$$

$$\frac{198}{140} \times \frac{60}{99} = \frac{2 \times 3 \times 3 \times 11}{2 \times 2 \times 5 \times 7} \times \frac{2 \times 2 \times 5 \times 3}{3 \times 3 \times 11} = \frac{\cancel{2} \times \cancel{3} \times \cancel{3} \times \cancel{11}}{\cancel{2} \times \cancel{2} \times \cancel{5} \times 7} \times \frac{\cancel{2} \times 2 \times \cancel{5} \times 3}{\cancel{3} \times \cancel{3} \times \cancel{11}} = \frac{6}{7}$$

9.3 Phrasing

$\frac{1}{3} \times \frac{1}{4}$ is read as "one third times one fourth" or "one third of one fourth."

We do not say "one third of a fourth" or "one third of the fourth" because we are not referring to a particular fourth, but to the number $\frac{1}{4}$.

In doing so, we transfer the phrasing that we know from whole numbers to the multiplication of fractions, because:

3×4 is "three times four", not "three times a four", or "three times the four", or "three times one of the fours." Similarly, we describe

$\frac{2}{5} \times \frac{9}{7}$ as "two fifths times nine sevenths" or "two fifths of nine sevenths".

We do not say "two fifths of nine times one seventh".

9.4 Explaining Fraction Multiplication

We know multiplication of whole numbers as a shortcut for repeated addition, e.g.:

$$3 \times 4 = 4 + 4 + 4 = 12 \ .$$

So, 3×4 is three times four.

When we multiply a whole number by a fraction, we want to understand it the same way.

Then, for example, $3 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$. So $3 \times \frac{1}{4}$ is three times one fourth.

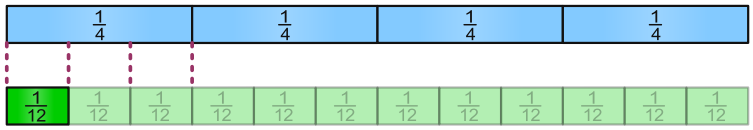
We arrive at the same result if we use the definition of fraction multiplication given earlier:

$$3 \times \frac{1}{4} = \frac{3}{1} \times \frac{1}{4} = \frac{3 \times 1}{1 \times 4} = \frac{3}{4}$$

What, then, could $\frac{1}{3} \times \frac{1}{4}$ be?

If $3 \times \frac{1}{4}$ is three times one fourth, then $\frac{1}{3} \times \frac{1}{4}$ could be one third of one fourth.

As we can see, e.g., by looking at the fraction strips, one third of one fourth is one twelfth, because if we divide each fourth into three equal parts, then $3 \times 4 = 12$ of those parts fit into one whole.



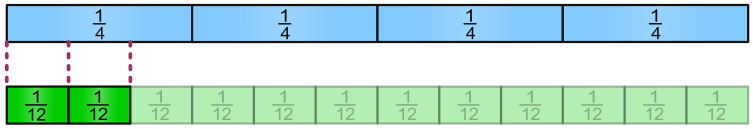
We arrive at the same result if we multiply *numerator times numerator* and *denominator times denominator*.

$$\frac{1}{3} \times \frac{1}{4} = \frac{1 \times 1}{3 \times 4} = \frac{1}{12}$$

From the natural numbers, we know the following situation: When we calculate, for example, 3×4, we get 12, and when we calculate 6×4, we get *twice* as much as 12, namely 24. Accordingly, the result of $\frac{2}{3} \times \frac{1}{4}$ should be *twice* as large as the result of $\frac{1}{3} \times \frac{1}{4}$.
We obtain exactly this result when we use the definition:

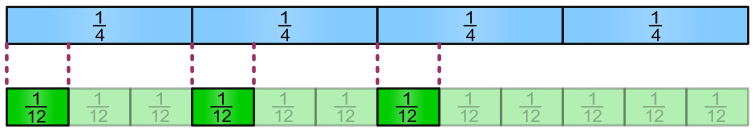
$$\frac{2}{3} \times \frac{1}{4} = \frac{2 \times 1}{3 \times 4} = \frac{2}{12}$$

We can also verify this using fraction strips. When we take one third of one fourth, we get one twelfth. When we take *two* thirds of one fourth, we get *two* twelfths.



To illustrate $\frac{1}{3} \times \frac{3}{4}$ with the fraction strips, we could divide $\frac{3}{4}$ into 3 equal parts. That would be correct and reasonable, but then we would not be able to see so clearly that the rule *numerator times numerator and denominator times denominator* is meaningful.

Therefore, we choose the following method: We take one third of each of the 3 fourths of the fraction strip. As we can see, that is $\frac{3}{12}$.



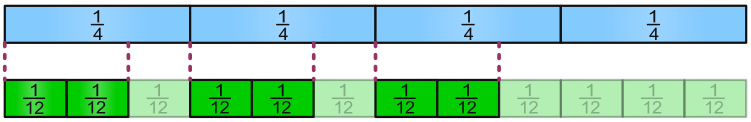
If we multiply according to the definition, exactly the same result occurs:

$$\frac{1}{3} \times \frac{3}{4} = \frac{1 \times 3}{3 \times 4} = \frac{3}{12}$$

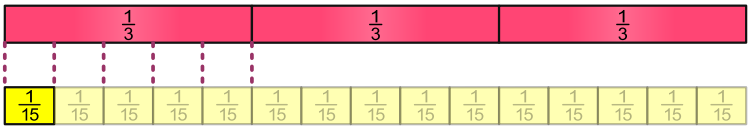
Following this logic, $\frac{2}{3} \times \frac{3}{4}$ should be three times as large as $\frac{2}{3} \times \frac{1}{4}$.
We get this result by using the definition:

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$$

since six twelfths are three times as large as two twelfths.



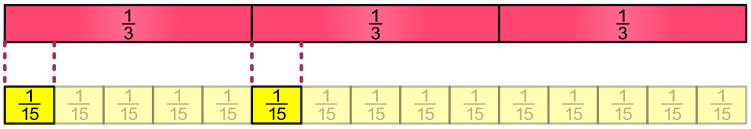
Let us examine fraction multiplication with another example:
If we divide one third into 5 equal parts, each of these parts is one fifteenth, because $5 \times 3 = 15$ of these parts fit into one whole.



The result of $\frac{1}{5} \times \frac{1}{3}$ answers the question: "How much is $\frac{1}{5}$ of $\frac{1}{3}$?"
We calculate:

$$\frac{1}{5} \times \frac{1}{3} = \frac{1 \times 1}{5 \times 3} = \frac{1}{15}$$

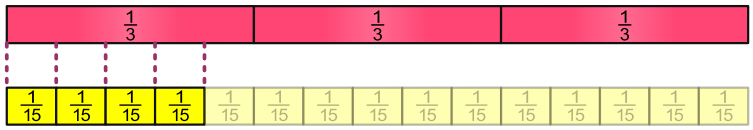
The result of $\frac{1}{5} \times \frac{2}{3}$ answers the question: "How much is $\frac{1}{5}$ of $\frac{2}{3}$?"



We calculate:

$$\frac{1}{5} \times \frac{2}{3} = \frac{1 \times 2}{5 \times 3} = \frac{2}{15}$$

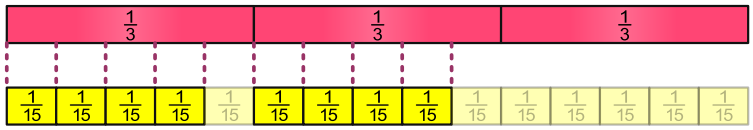
The result of $\frac{4}{5} \times \frac{1}{3}$ answers the question: "How much is $\frac{4}{5}$ of $\frac{1}{3}$?"



We calculate:

$$\frac{4}{5} \times \frac{1}{3} = \frac{4 \times 1}{5 \times 3} = \frac{4}{15}$$

The result of $\frac{4}{5} \times \frac{2}{3}$ answers the question: "How much is $\frac{4}{5}$ of $\frac{2}{3}$?"



We calculate:

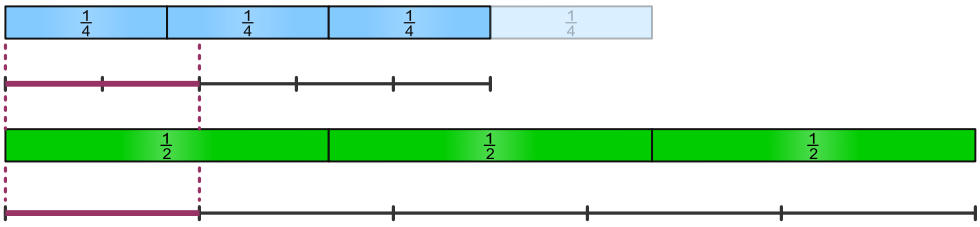
$$\frac{4}{5} \times \frac{2}{3} = \frac{4 \times 2}{5 \times 3} = \frac{8}{15}$$

9.5 Cross-Canceling

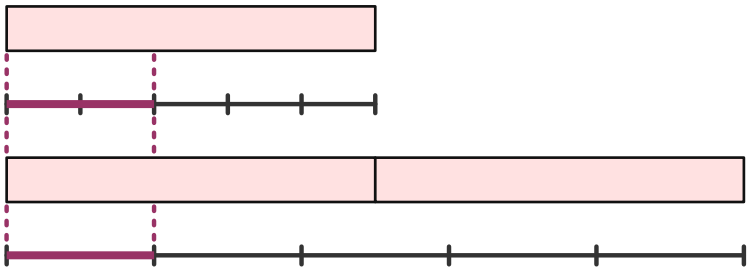
When we want to find $\frac{2}{5} \times \frac{3}{4}$, we simplify before we multiply. So:

$$\frac{2}{5} \times \frac{3}{4} = \frac{2}{5} \times \frac{3}{2 \times 2} = \frac{\cancel{2}}{5} \times \frac{3}{\cancel{2} \times 2} = \frac{1}{5} \times \frac{3}{2}$$

But why does this work? Why is it allowed to simplify "across" like that?
Let's examine this relationship using fraction strips: as we can see, $\frac{3}{2}$ is twice as large as $\frac{3}{4}$.
So instead of taking *two* fifths of $\frac{3}{4}$, we only take *one* fifth of a number that is twice as large as $\frac{3}{4}$, namely $\frac{3}{2}$.



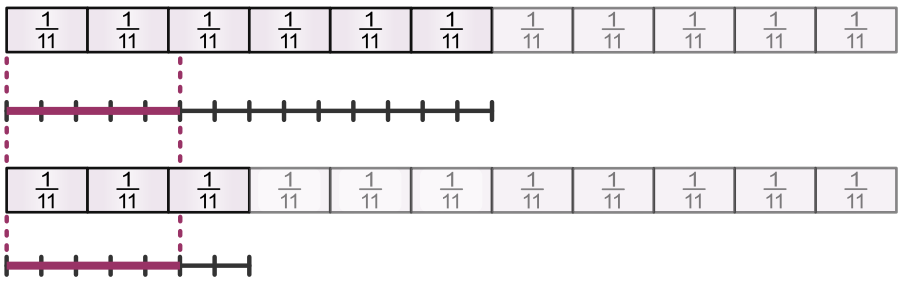
We can also understand this in a more general way: Suppose we are given some length, and we want to take $\frac{2}{5}$ of it. Then we could just as well take $\frac{1}{5}$ of double that length, and we would get the same result.



More Examples

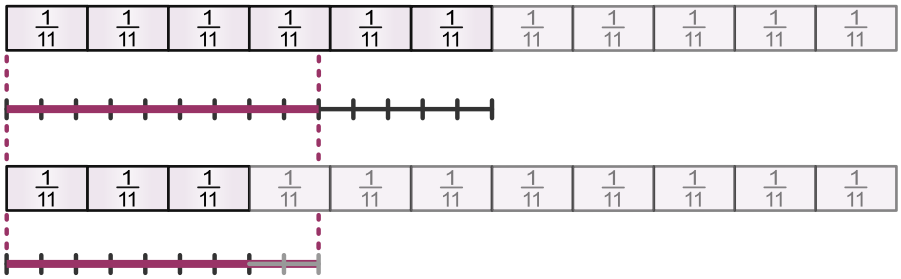
$$\frac{5}{14} \times \frac{6}{11} = \frac{5}{7} \times \frac{3}{11}$$

We divide one bar into 14 parts. And we divide a bar that is half as long into 7 parts. As we can see, the sevenths of the shorter bar are the same length as the fourteenths of the full-length bar.



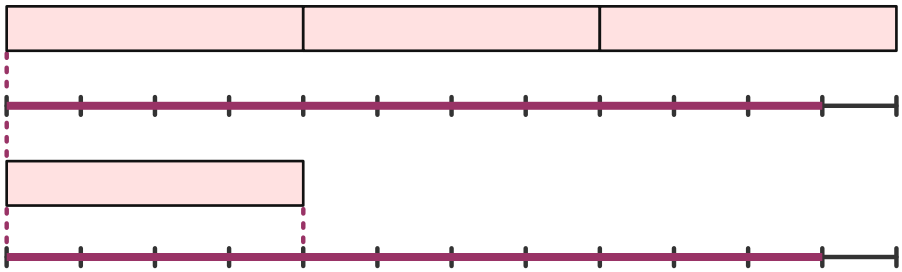
This kind of representation even works when simplifying makes the numerator of the first fraction greater than its denominator.

$$\frac{9}{14} \times \frac{6}{11} = \frac{9}{7} \times \frac{3}{11}$$



In this example, we are working with a segment of arbitrary length a. The twelfths of three times a are the same length as the fourths of the unit-length segment a.

$$\frac{11}{12} \times 3a = \frac{11}{4}a$$



9.6 Swapping Numerators and Denominators

When we multiply two fractions, we can swap the numerators without changing the result. The same applies to the denominators. For example:

$$\frac{2}{3} \times \frac{5}{7} = \frac{5}{3} \times \frac{2}{7} = \frac{5}{7} \times \frac{2}{3} = \frac{2}{7} \times \frac{5}{3}$$

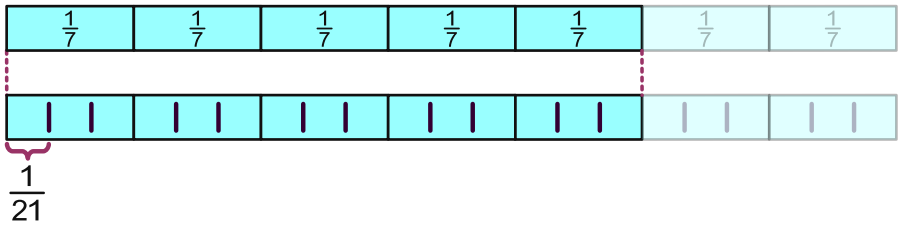
Why does it work?

Original Fractions

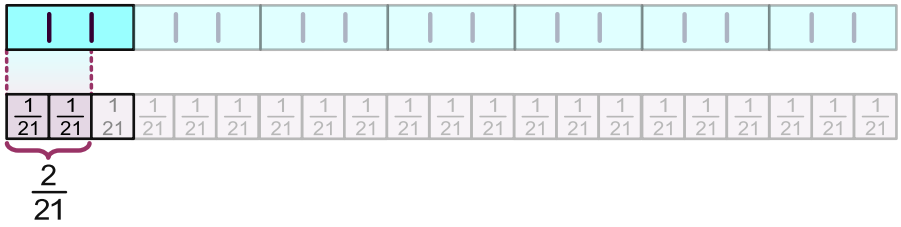
Let's take a look at how we multiply $\frac{2}{3} \times \frac{5}{7}$:

First, we divide all 7 sevenths into 3 equal units. This gives us $3 \times 7 = 21$ units in total.

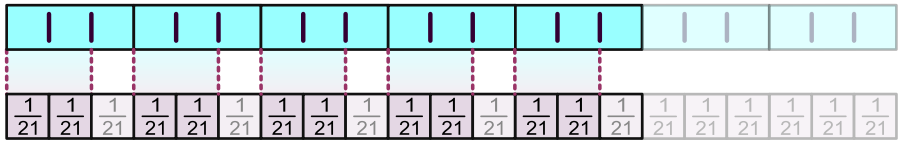
So one of these units is $\frac{1}{21}$ of the whole.



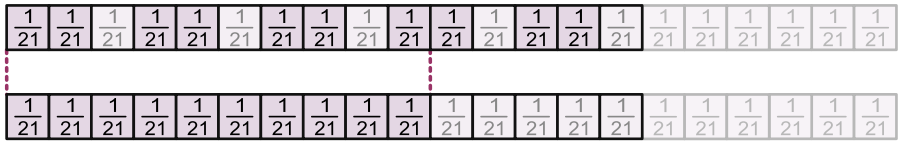
$\frac{2}{3}$ of $\frac{1}{7}$ are *two* units of one part of the fraction, i.e., $\frac{2}{21}$.



$\frac{2}{3}$ of $\frac{5}{7}$ are *two* units of *five* parts, so we get $2 \times 5 = 10$ units in total.

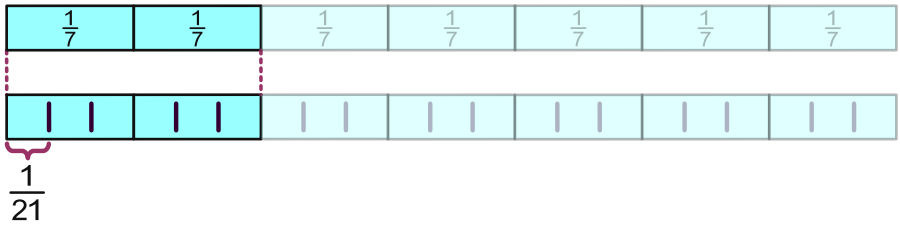


The result is $\frac{10}{21}$.

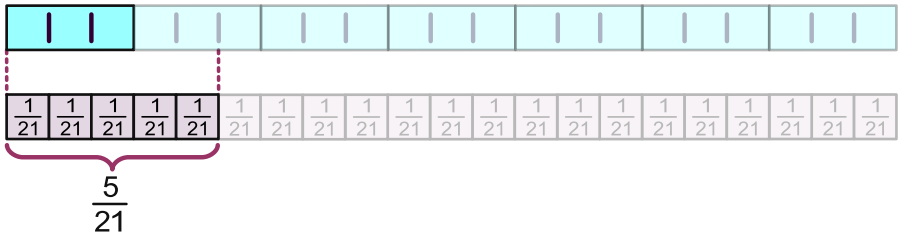


Swapped Numerators

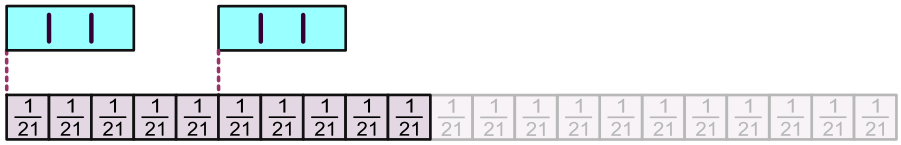
Let's now look at $\frac{5}{3} \times \frac{2}{7}$:
First, we divide all 7 sevenths into 3 units. This gives $3 \times 7 = 21$ units in total.
So each unit represents $\frac{1}{21}$ of the whole.



$\frac{5}{3}$ of $\frac{1}{7}$ are *five* units of *one* part of the fraction, so this gives $\frac{5}{21}$.



$\frac{5}{3}$ of $\frac{2}{7}$ are *five* units of *two* parts, which gives $5 \times 2 = 10$ units in total.

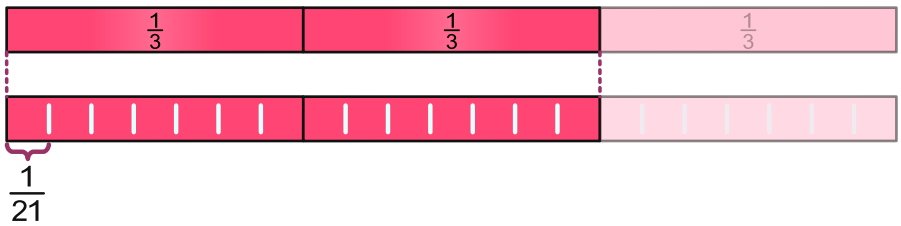


So the result is $\frac{10}{21}$.

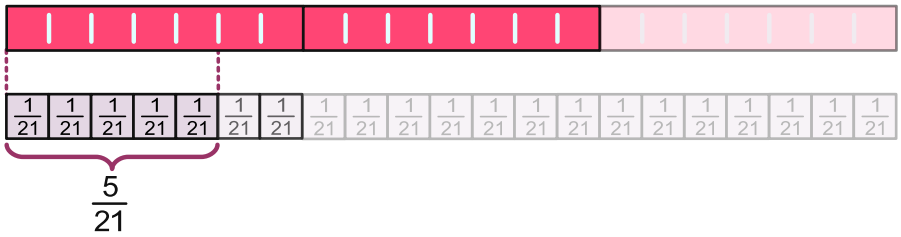
Why does the result stay the same?
Since the denominators stayed the same, the units stayed the same too — they are twenty-firsts.
The number of units in the result also stayed the same: In the first case, we took *two* units of *five* parts; in the second case, we took *five* units of *two* parts. In both cases, the result is *10 units*.

Swapping Denominators

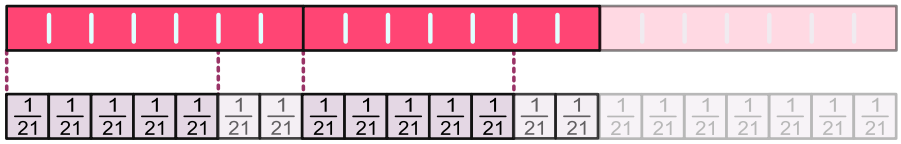
Let's now look at $\frac{5}{7} \times \frac{2}{3}$:
First, we divide each of the 3 thirds into 7 units. This gives us a total of $7 \times 3 = 21$ units.
Each unit is therefore $\frac{1}{21}$ of the whole.



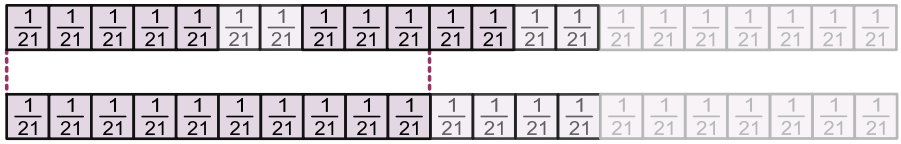
$\frac{5}{7}$ of $\frac{1}{3}$ are *five* units of *one* part of the fraction, so the result is $\frac{5}{21}$.



$\frac{5}{7}$ of $\frac{2}{3}$ means *five* units of *two* parts, which gives us a total of $5 \times 2 = 10$ units.



So the result is $\frac{10}{21}$.



Why does $\frac{5}{7} \times \frac{2}{3}$ give the same result as $\frac{5}{3} \times \frac{2}{7}$?

Since the numerators remain the same, we are again taking *five* units of *two* parts, which gives us 10 units.

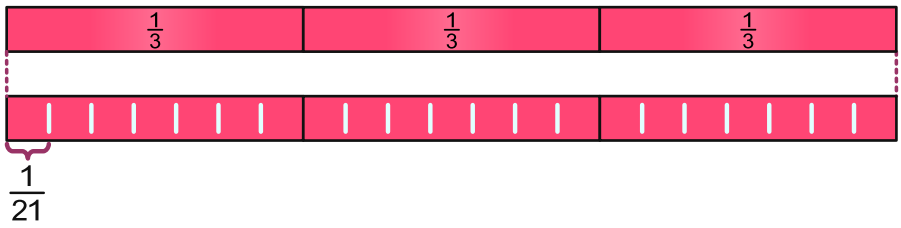
In the previous calculation, we divided 7 sevenths into 3 units each, resulting in 21-sths. In this calculation, we divide 3 thirds into 7 units each— again resulting in 21-sths. That's why the denominator stays the same.

Swapped Numerators (Again)

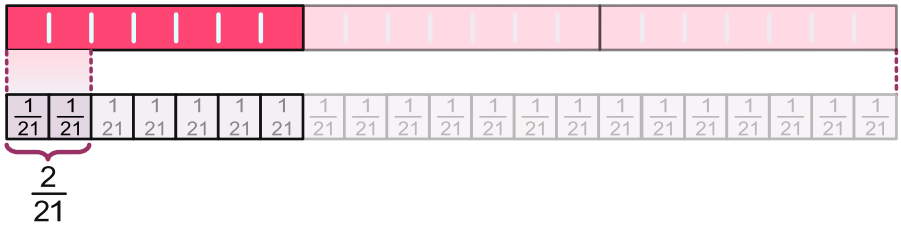
Finally, we have $\frac{2}{7} \times \frac{5}{3}$:

First, we divide each of the 3 thirds into 7 units. This gives us $7 \cdot 3 = 21$ units in total.

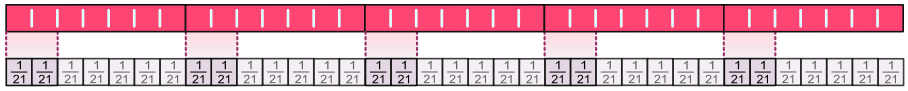
So one unit is $\frac{1}{21}$ of the whole.



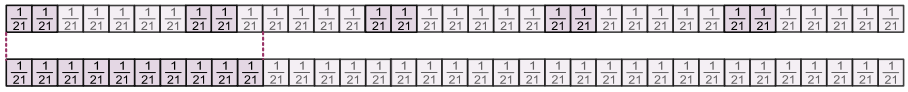
$\frac{2}{7}$ of $\frac{1}{3}$ are *two* units of *one* part of the fraction, which gives us $\frac{2}{21}$.



$\frac{2}{7}$ of $\frac{5}{3}$ means we take *two* units of *five* parts, which gives us $2 \times 5 = 10$ units.



So the result is $\frac{10}{21}$.



Why do we get the same result from $\frac{2}{7} \times \frac{5}{3}$ as from all the previous products? The numerator stays the same because we are either taking *five* units of *two* parts or *two* units of *five* parts. The total number of units is the same in both cases. The denominator stays the same because we are either dividing 3 thirds into 7 units each or dividing 7 sevenths into 3 units each. The total number of units is the same in both cases.

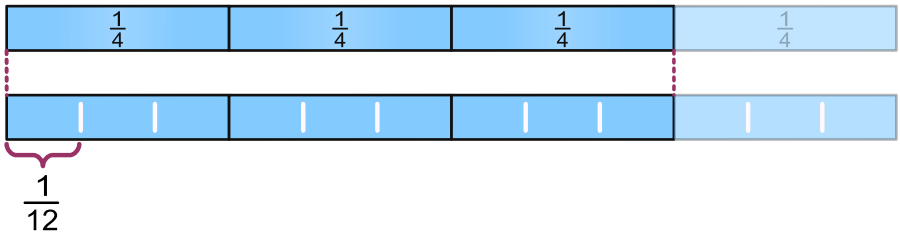
9.7 Examples

Let's take a look at a few more examples:

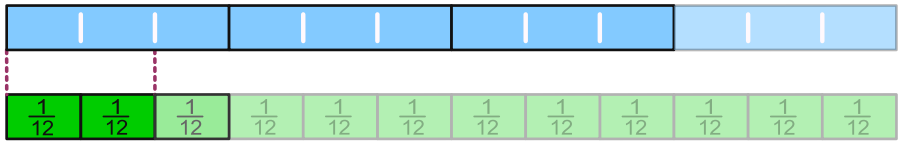
Example 1

We want to multiply the fractions $\frac{2}{3}$ and $\frac{3}{4}$, that is, we want to calculate $\frac{2}{3} \times \frac{3}{4}$. We ask ourselves: What are two thirds of three fourths?

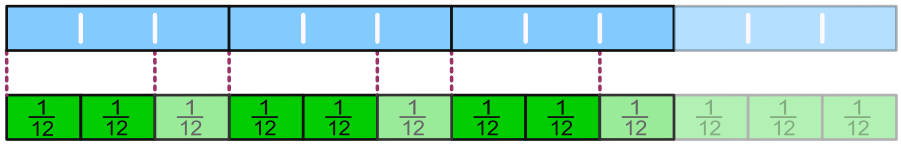
First, we divide each of the 4 fourths into 3 units. This gives us $3 \times 4 = 12$ units. So one unit is $\frac{1}{12}$ of the whole.



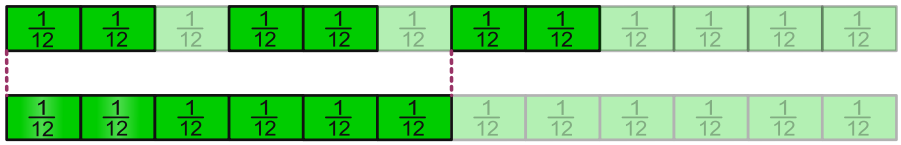
$\frac{2}{3}$ of $\frac{1}{4}$ are 2 units.



$\frac{2}{3}$ of $\frac{3}{4}$ are then $2 \times 3 = 6$ units.

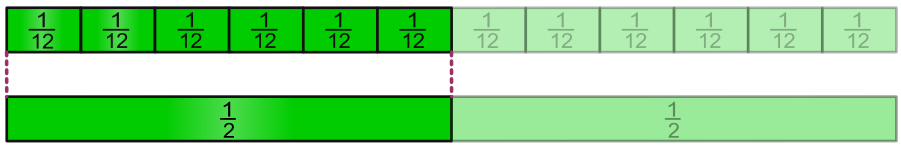


Put together, these are $\frac{6}{12}$.



Because we simplify the result, we now get the following calculation:

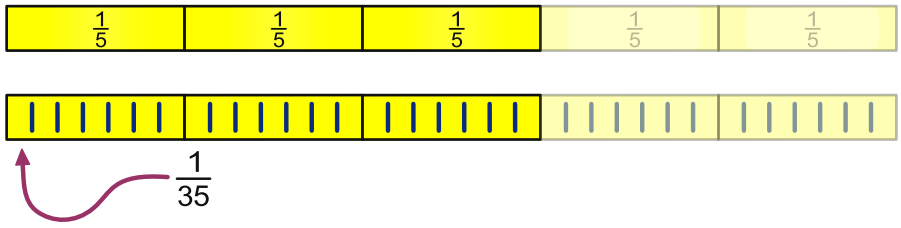
$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2}$$



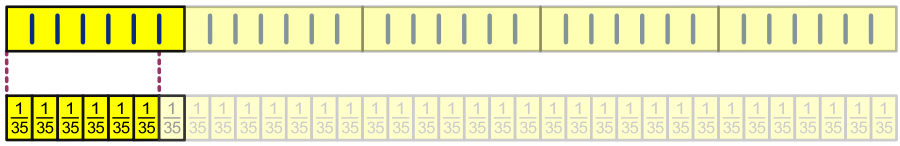
Example 2

We want to multiply the fractions $\frac{6}{7}$ and $\frac{3}{5}$, so we compute $\frac{6}{7} \times \frac{3}{5}$.
We ask ourselves: What are six sevenths of three fifths?

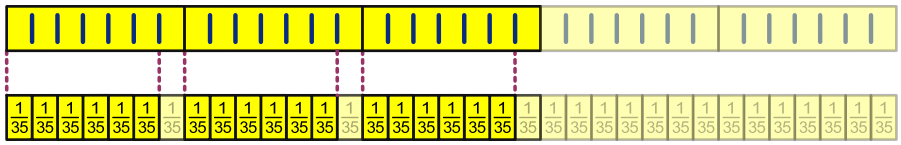
First, we divide all 5 fifths into 7 units. That gives us $7 \times 5 = 35$ units in total. So one unit is equal to $\frac{1}{35}$ of the whole.



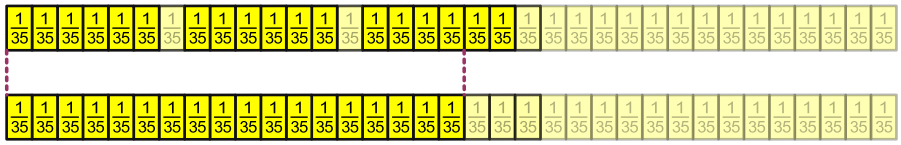
$\frac{6}{7}$ of $\frac{1}{5}$ are 6 units.



$\frac{6}{7}$ of $\frac{3}{5}$ means we take *6* units of *3* parts. That gives us $6 \times 3 = 18$ units.



Put together, that gives us $\frac{18}{35}$.



Here is the full calculation:

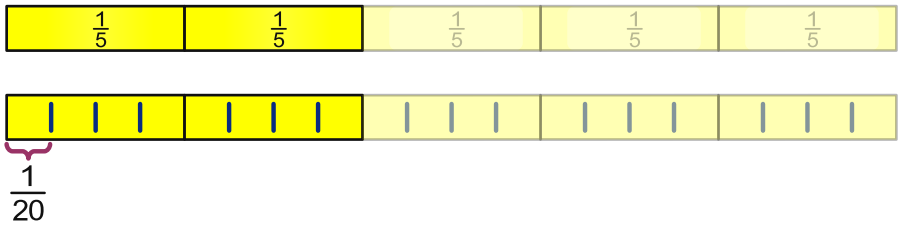
$$\frac{6}{7} \times \frac{3}{5} = \frac{6 \times 3}{7 \times 5} = \frac{18}{35}$$

Example 3

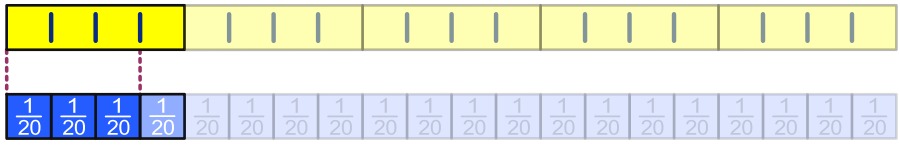
We want to multiply the fractions $\frac{3}{4}$ and $\frac{2}{5}$, so we compute $\frac{3}{4} \times \frac{2}{5}$.

We ask ourselves: What are three fourths of two fifths?

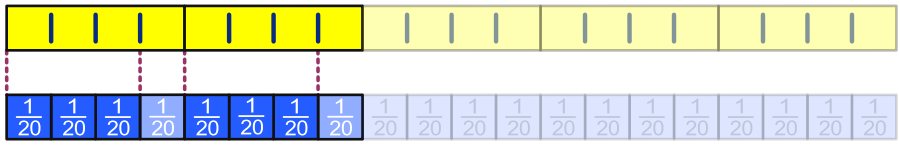
First, we divide all 5 fifths into 4 units. That gives us $4 \times 5 = 20$ units. So one unit is $\frac{1}{20}$ of the whole.



$\frac{3}{4}$ of $\frac{1}{5}$ are 3 units.



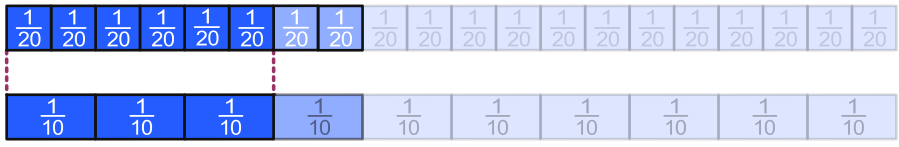
$\frac{3}{4}$ of $\frac{2}{5}$ are *3* units of *2* parts. That gives us $3 \times 2 = 6$ units.



Put together, that gives us $\frac{6}{20}$.

Since we simplify the result, we now get the following calculation:

$$\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20} = \frac{6 \div 2}{20 \div 2} = \frac{3}{10}$$

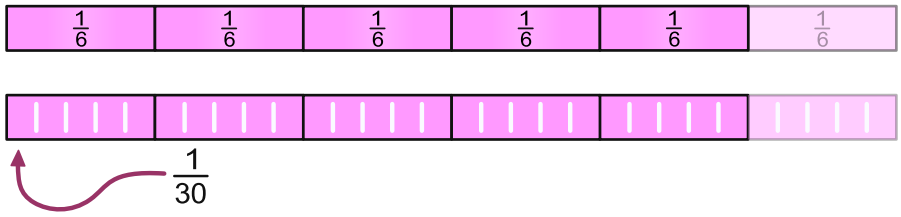


Example 4

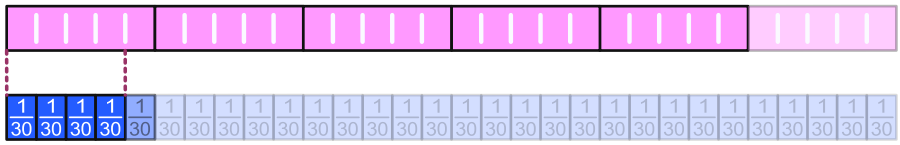
We want to multiply the fractions $\frac{4}{5}$ and $\frac{5}{6}$, so we compute $\frac{4}{5} \times \frac{5}{6}$.

Let's ask ourselves: What are four fifths of five sixths?

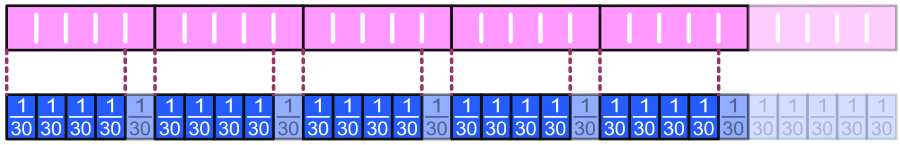
First, we divide all 6 sixths into 5 units. That gives us $5 \times 6 = 30$ units. So one unit is $\frac{1}{30}$ of the whole.



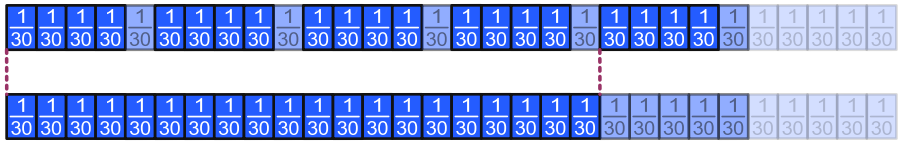
$\frac{4}{5}$ of $\frac{1}{6}$ are 4 units.



$\frac{4}{5}$ of $\frac{5}{6}$ are *4* units of *5* parts. That gives us $4 \times 5 = 20$ units.

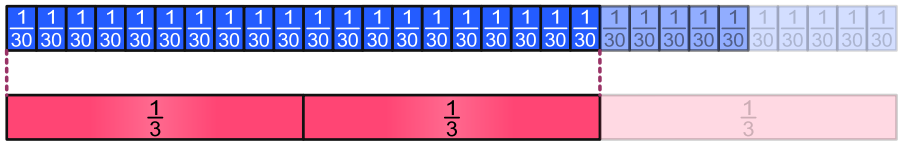


Put together, that gives us $\frac{20}{30}$.



Since we simplify the result, we now get the following calculation:

$$\frac{4}{5} \times \frac{5}{6} = \frac{4 \times 5}{5 \times 6} = \frac{20}{30} = \frac{20 \div 10}{30 \div 10} = \frac{2}{3}$$



By the way, we would have gotten the same result if we had simplified the fractions *before* multiplying. But then, there would have been nothing left to multiply.

$$\frac{4}{5} \times \frac{5}{6} = \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{6} = \frac{4}{1} \times \frac{1}{6} = \frac{2 \times 2}{1} \times \frac{1}{2 \times 3} = \frac{\cancel{2} \times 2}{1} \times \frac{1}{\cancel{2} \times 3} = \frac{2}{1} \times \frac{1}{3} = \frac{2}{3}$$

10 Dividing Fractions

10.1 Definition

In mathematics, the following rule has become standard:

> To divide by a fraction,
> multiply by the reciprocal.

Using variables, we can write this very succinctly:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$$

Here, a, b, c, and d are natural numbers, with b and d greater than 0.

Examples

$\frac{1}{2} \div \frac{1}{3} = \frac{1 \times 3}{2 \times 1} = \frac{3}{2}$

Unlike division of natural numbers, the result can be greater than the number being divided.

$\frac{2}{3} \div \frac{4}{5} = \frac{2 \times 5}{3 \times 4} = \frac{10}{12} = \frac{5}{6}$

The fraction that is being divided is called the **dividend**. The fraction by which we divide is called the **divisor**.

$\frac{4}{1} \div \frac{6}{1} = \frac{4 \times 1}{1 \times 6} = \frac{4}{6} = \frac{2}{3}$

If we apply fraction division to natural numbers, we obtain the results we are familiar with from arithmetic with natural numbers, because $\frac{4}{1}$ equals 4, $\frac{6}{1}$ equals 6, and the result of $4 \div 6$ is $\frac{2}{3}$.

10.2 Explaining the Division by Fractions

What we humans want to understand by the division by fractions is something we must decide for ourselves. There is nothing in nature that forces us to assign any particular meaning to the division of fractions. We now look at one way to make sense of the definition given above.

One way to consider what division by fractions might mean is to extend the properties of division of natural numbers into the realm of fractions. This approach will be pursued further below.

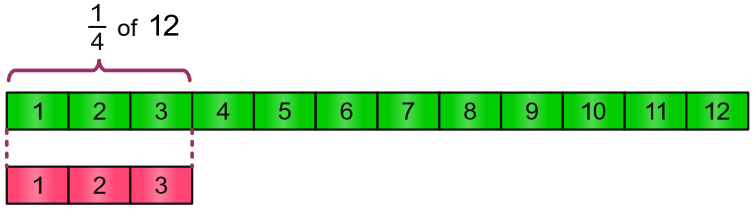
When we divide 12 by 3, we get 4 because 3 fits into 12 four times.
When we divide 3 by 12, we do not ask how many times 12 fits into 3, instead, we ask:

- Which part of 12 fits into 3?

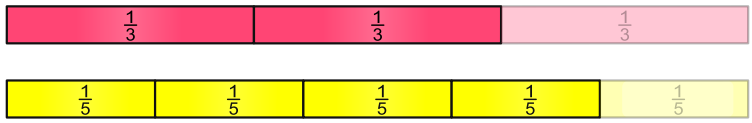
or alternatively:

- How much of 12 fits into 3?

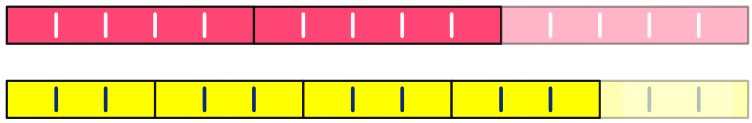
The answer is $\frac{1}{4}$, because one quarter of 12 equals 3.



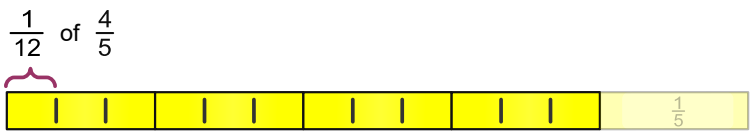
When we divide fractions, we ask the same question. For example, we can formulate the problem $\frac{2}{3} \div \frac{4}{5}$ as: Which part of $\frac{4}{5}$ fits into $\frac{2}{3}$?
To help visualize the calculation, we lay out fraction strips.



To answer the question of how much of $\frac{4}{5}$ fits into $\frac{2}{3}$, we divide the fractions into smaller units. We divide each third into 5 parts, and each fifth into 3 parts.
All the small units are now equal in size.

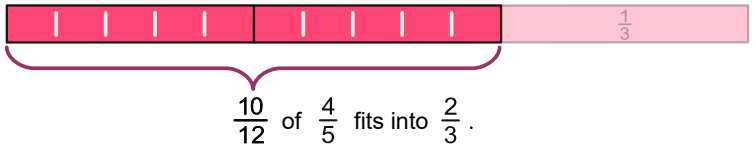


On the 4 fifths, there are $3 \times 4 = 12$ units.
Therefore, each unit is $\frac{1}{12}$ of $\frac{4}{5}$.
To obtain the denominator 12 of the result, we multiply the denominator of the dividend by the numerator of the divisor. This is one part of the reciprocal rule.



On the 2 thirds, there are $2 \times 5 = 10$ units.
Therefore $\frac{10}{12}$ of $\frac{4}{5}$ fits into $\frac{2}{3}$.

To obtain the numerator 10 of the result, we multiply the numerator of the dividend by the denominator of the divisor. This is the other part of the reciprocal rule.



We write the calculation as follows:

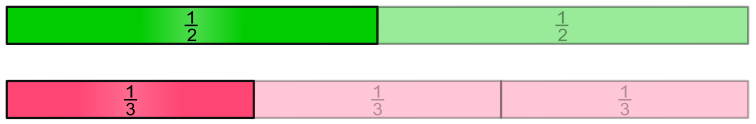
$$\frac{2}{3} \div \frac{4}{5} = \frac{2 \times 5}{3 \times 4} = \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

So, as a result, $\frac{5}{6}$ of $\frac{4}{5}$ fits into $\frac{2}{3}$.

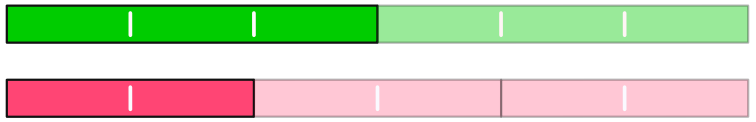
10.3 Examples

Example 1

What is $\frac{1}{2} \div \frac{1}{3}$? In other words: How much of $\frac{1}{3}$ fits into $\frac{1}{2}$?

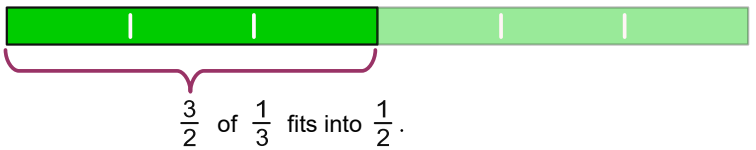


We divide one half into 3 units, and one third into 2 units.



We have divided one third into $2 \times 1 = 2$ units. So each unit is $\frac{1}{2}$ of $\frac{1}{3}$.

We have divided one half into $1 \times 3 = 3$ units. So $\frac{3}{2}$ of $\frac{1}{3}$ fits into $\frac{1}{2}$.

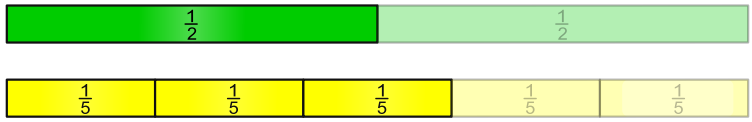


The calculation:

$$\frac{1}{2} \div \frac{1}{3} = \frac{1 \times 3}{2 \times 1} = \frac{3}{2}$$

Example 2

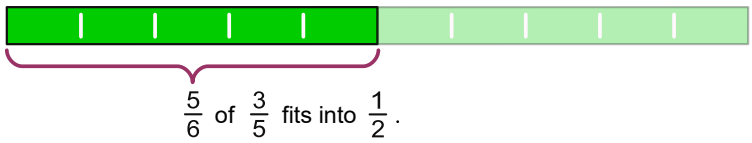
What is $\frac{1}{2} \div \frac{3}{5}$? In other words: How much of $\frac{3}{5}$ fits into $\frac{1}{2}$?



We divide one half into 5 units, and each fifth into 2 units.

We have divided the fifths into $2 \times 3 = 6$ units. Therefore, each unit is $\frac{1}{6}$ of $\frac{3}{5}$.

We have divided one half into $1 \times 5 = 5$ units. Therefore $\frac{5}{6}$ of $\frac{3}{5}$ fits into $\frac{1}{2}$.

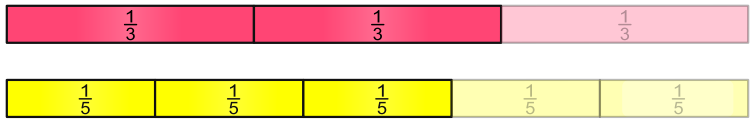


The calculation:

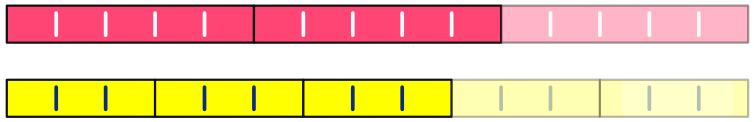
$$\frac{1}{2} \div \frac{3}{5} = \frac{1 \times 5}{2 \times 3} = \frac{5}{6}$$

Example 3

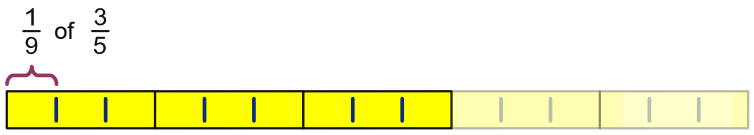
What is $\frac{2}{3} \div \frac{3}{5}$? In other words: How much of $\frac{3}{5}$ fits into $\frac{2}{3}$?



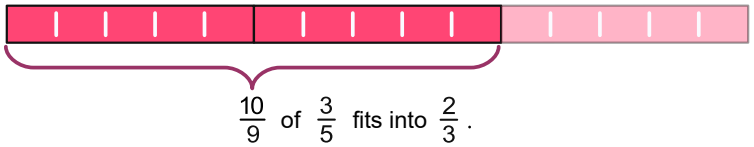
We divide each third into 5 units, and each fifth into 3 units.



We have divided the fifths into $3 \times 3 = 9$ units. Therefore each unit is $\frac{1}{9}$ of $\frac{3}{5}$.



We have divided the thirds into $2 \times 5 = 10$ units. Therefore, $\frac{10}{9}$ of $\frac{3}{5}$ fits into $\frac{2}{3}$.

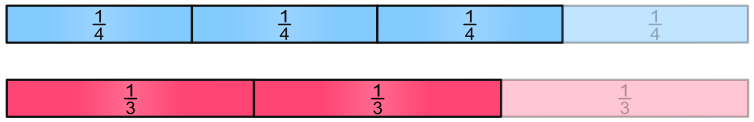


The calculation:

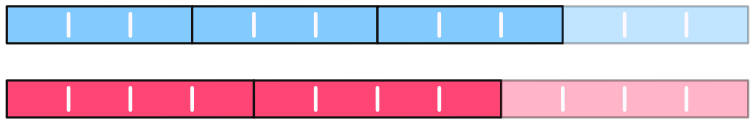
$$\frac{2}{3} \div \frac{3}{5} = \frac{2 \times 5}{3 \times 3} = \frac{10}{9}$$

Example 4

What is $\frac{3}{4} \div \frac{2}{3}$? In other words: How much of $\frac{2}{3}$ fits into $\frac{3}{4}$?

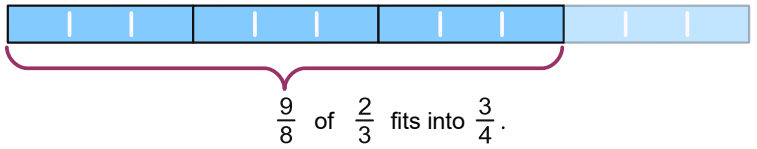


We divide each fourth into 3 units, and each third into 4 units.



We have divided the thirds into $4 \times 2 = 8$ units. Therefore, each unit is $\frac{1}{8}$ of $\frac{2}{3}$.

We have divided the fourths into $3 \times 3 = 9$ units. Therefore, $\frac{9}{8}$ of $\frac{2}{3}$ fits into $\frac{3}{4}$.

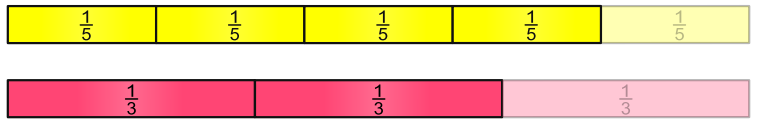


The calculation:

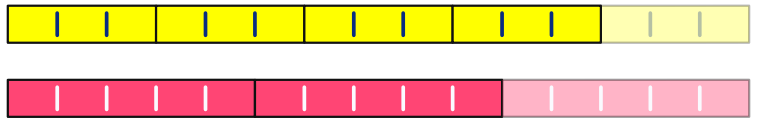
$$\frac{3}{4} \div \frac{2}{3} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8}$$

Example 5

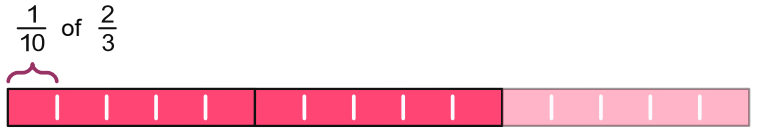
What is $\frac{4}{5} \div \frac{2}{3}$? In other words: How much of $\frac{2}{3}$ fits into $\frac{4}{5}$?



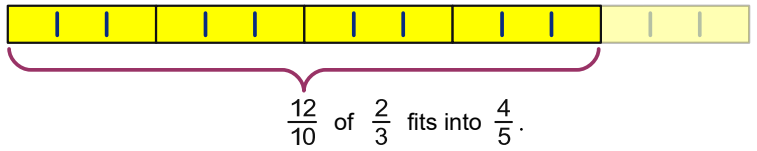
We divide each fifth into 3 units, and each third into 5 units.



We have divided the thirds into $5 \times 2 = 10$ units. Therefore, each unit is $\frac{1}{10}$ of $\frac{2}{3}$.



We have divided the fifths into $4 \times 3 = 12$ units. Therefore, $\frac{12}{10}$ of $\frac{2}{3}$ fits into $\frac{4}{5}$.

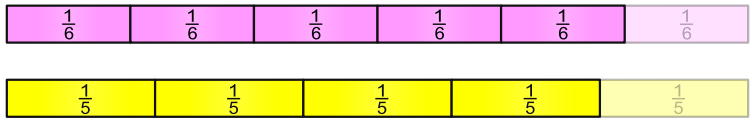


The calculation:

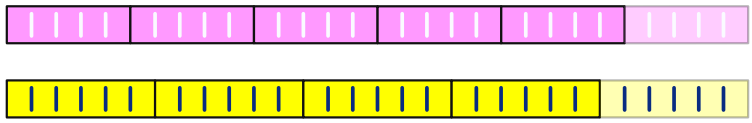
$$\frac{4}{5} \div \frac{2}{3} = \frac{4 \times 3}{5 \times 2} = \frac{12}{10} = \frac{12 \div 2}{10 \div 2} = \frac{6}{5}$$

Example 6

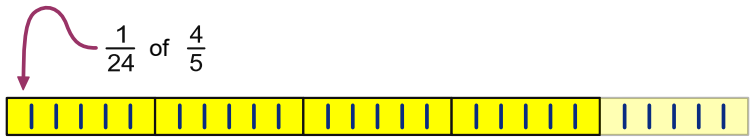
What is $\frac{5}{6} \div \frac{4}{5}$? In other words: How much of $\frac{4}{5}$ fits into $\frac{5}{6}$?



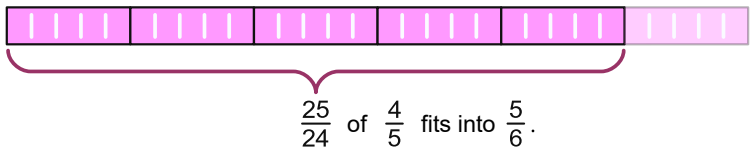
We divide each sixth into 5 units, and each fifth into 6 units.



We have divided the fifths into $6 \times 4 = 24$ units. Therefore, each unit is $\frac{1}{24}$ of $\frac{4}{5}$.



We have divided the sixths into $5 \times 5 = 25$ units. Therefore, $\frac{25}{24}$ of $\frac{4}{5}$ fits into $\frac{5}{6}$.

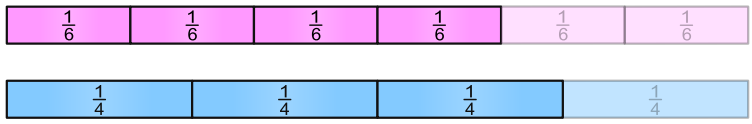


The calculation:

$$\frac{5}{6} \div \frac{4}{5} = \frac{5 \times 5}{6 \times 4} = \frac{25}{24}$$

Example 7

We want to divide $\frac{4}{6}$ by $\frac{3}{4}$.

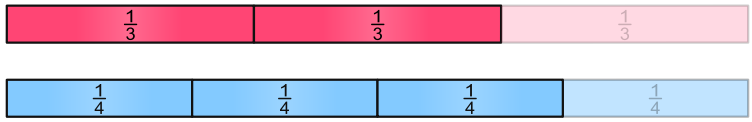


First, we simplify the fractions if possible. Thus:

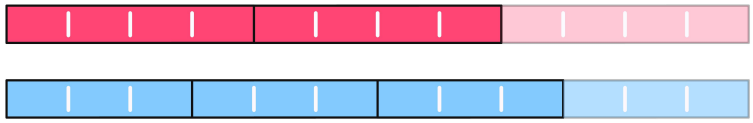
$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

The problem now is:

What is $\frac{2}{3} \div \frac{3}{4}$? In other words: How much of $\frac{3}{4}$ fits into $\frac{2}{3}$?

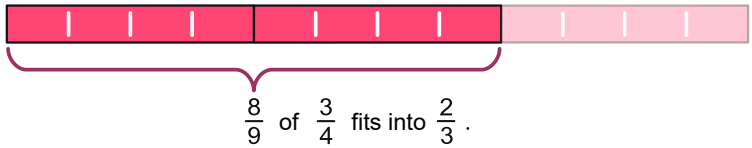


We divide each third into 4 units, and each fourth into 3 units.



We have divided the fourths into $3 \times 3 = 9$ units. Therefore, each unit is $\frac{1}{9}$ of $\frac{3}{4}$.

We have divided the thirds into $2 \times 4 = 8$ units. Therefore, $\frac{8}{9}$ of $\frac{3}{4}$ fits into $\frac{2}{3}$.

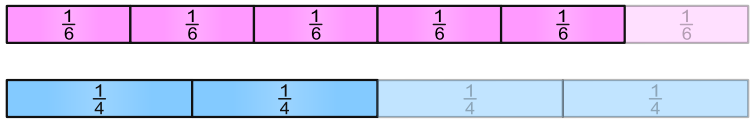


The calculation:

$$\frac{4}{6} \div \frac{3}{4} = \frac{4 \div 2}{6 \div 2} \div \frac{3}{4} = \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}$$

Example 8

We want to divide $\frac{5}{6}$ by $\frac{2}{4}$.

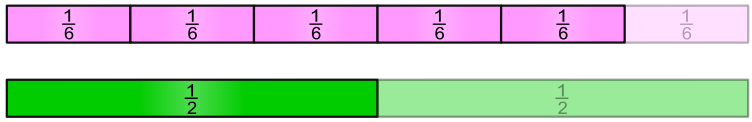


First, we simplify the fractions if possible. Thus:

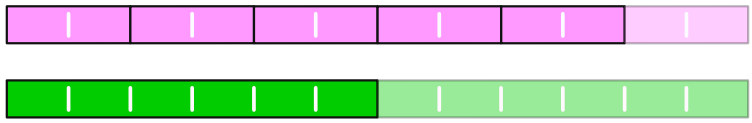
$$\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$$

The problem now is:

What is $\frac{5}{6} \div \frac{1}{2}$? In other words: How much of $\frac{1}{2}$ fits into $\frac{5}{6}$?

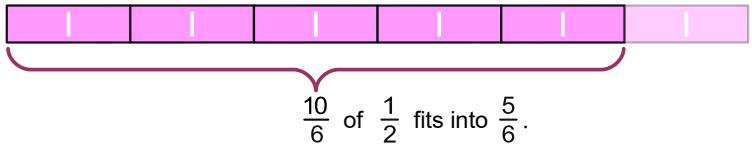


We divide each sixth into 2 units, and we divide one half into 6 units.



We have divided one half into $6 \times 1 = 6$ units. Therefore, each unit is $\frac{1}{6}$ of $\frac{1}{2}$.

We have divided the thirds into $5 \times 2 = 10$ units. Therefore, $\frac{10}{6}$ of $\frac{1}{2}$ fits into $\frac{5}{6}$.



The calculation:

$$\frac{5}{6} \div \frac{1}{2} = \frac{5 \times 2}{6 \times 1} = \frac{10}{6} = \frac{10 \div 2}{6 \div 2} = \frac{5}{3}$$