

Visual Explanation of the Fundamental Theorem of Calculus

To understand this article, you should know what derivatives and integrals are. You should also be familiar with slope triangles and area functions.

In order to make the intuitive ideas behind the Fundamental Theorem of Calculus as clear as possible, we will sacrifice some exactness and completeness in the presentation. For example, not all assumptions of the theorem will be stated, or certain relationships will be shown for a specific function without always mentioning for which class of functions these relationships also hold.

The Fundamental Theorem of Calculus is also called the Fundamental Theorem of Analysis. As these names suggest, this is a very important theorem. To understand what the statement of this theorem means, let's look at the function $\mathbf{f}(\mathbf{x})$ and its graph between $\mathbf{x}=\mathbf{0}$ and $\mathbf{x}=\mathbf{4}$, together with the corresponding area function $\mathbf{F}_0(\mathbf{x})$. The function value of $\mathbf{F}_0(\mathbf{x})$ at a certain point $\mathbf{x}$ equals the area under the graph of $\mathbf{f}(\mathbf{x})$ in the interval from 0 to $\mathbf{x}$.

For completeness, let's state exactly which functions we are talking about. They are:

$$f(x) = \frac{3}{5} \times x \left(\frac{1}{2} \times x - 2 \right)^2$$

and

$$F_0(x) = \frac{3}{80} \times x^4 - \frac{2}{5} \times x^3 + \frac{6}{5} \times x^2 .$$

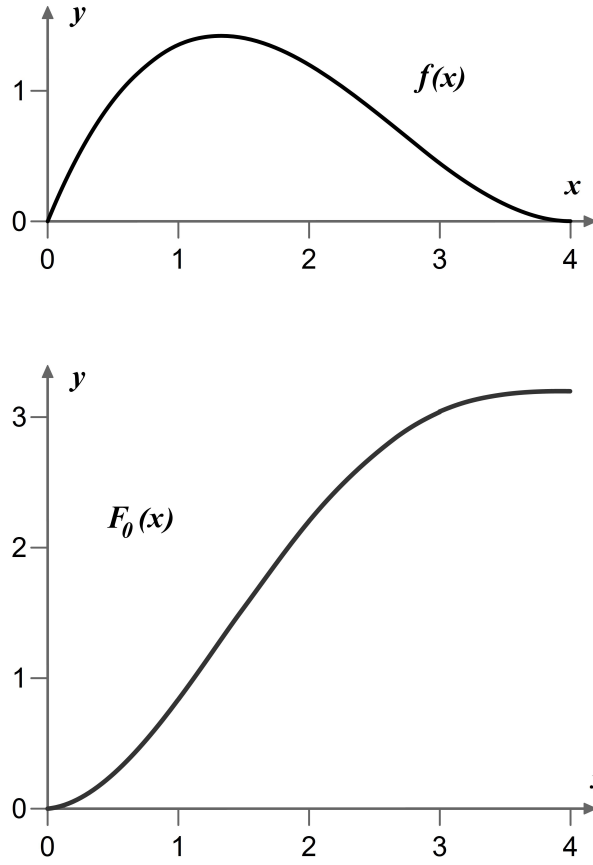


Fig. 1 Function and area function

The function value of $F_0(x)$ at **1** is, for example, **0.8375**, because the area of the blue region equals **0.8375** square units (SU).

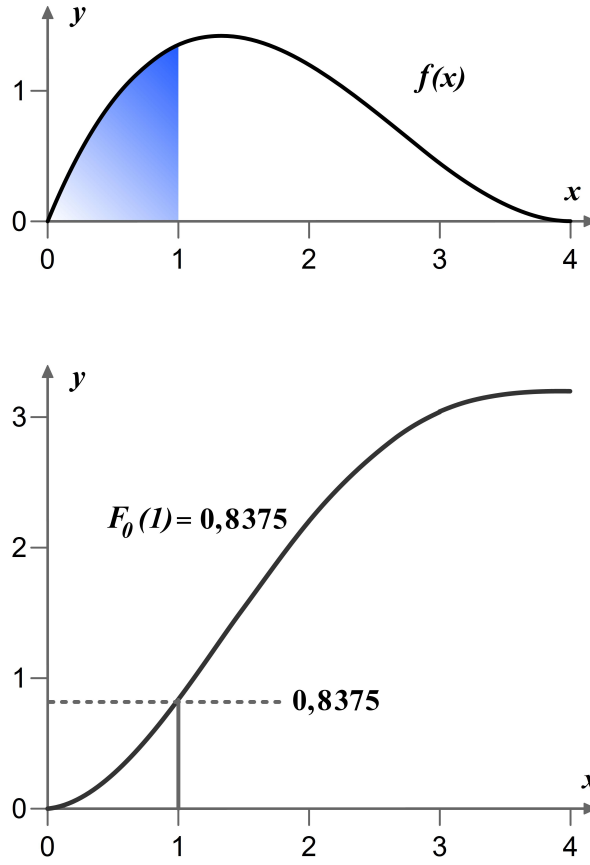


Fig. 2 Function and area function at $x = 1$

The function value of $F_0(x)$ at **2** is **2.2**, because the combined area of the blue and red regions equals **2.2** SU.

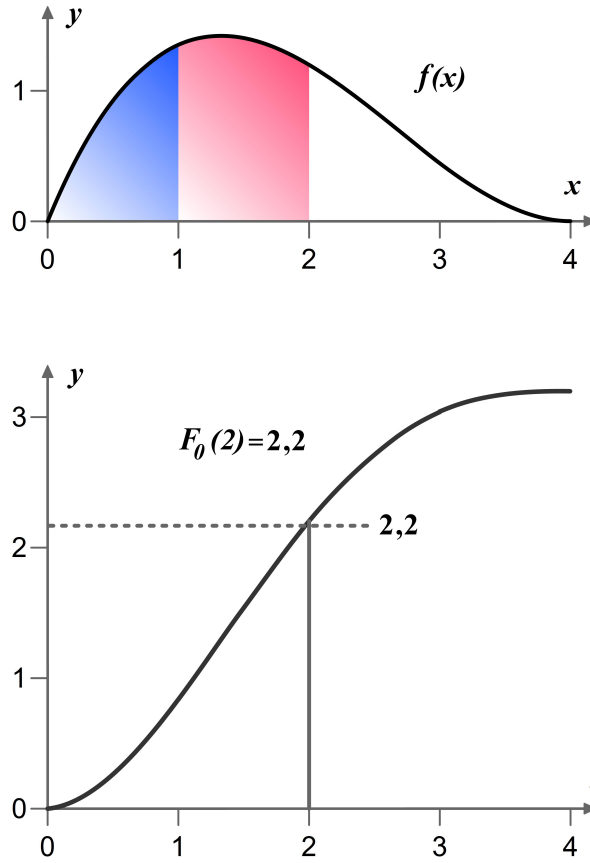


Fig. 3 Function and area function at $x = 2$

The theorem now states: The derivative of the area function $F_0(x)$ at any point x is equal to the function value of $f(x)$ at that point. In short:

$$(F_0(x))' = f(x)$$

This means: The slope of the graph of the area function at any point is equal to the function value of the function whose graph defines that area. That is quite surprising! After all, finding the area under a graph and finding the slope of a graph seem to have nothing to do with each other at first glance.

Let's take a closer look at what this means in our example: We pick some point, for instance $(3|F_0(3))$, and draw the tangent to the graph at that point. If we then draw a slope triangle whose horizontal side has length 1, the vertical side will have the same length as the function value of $f(x)$ at $x = 3$.

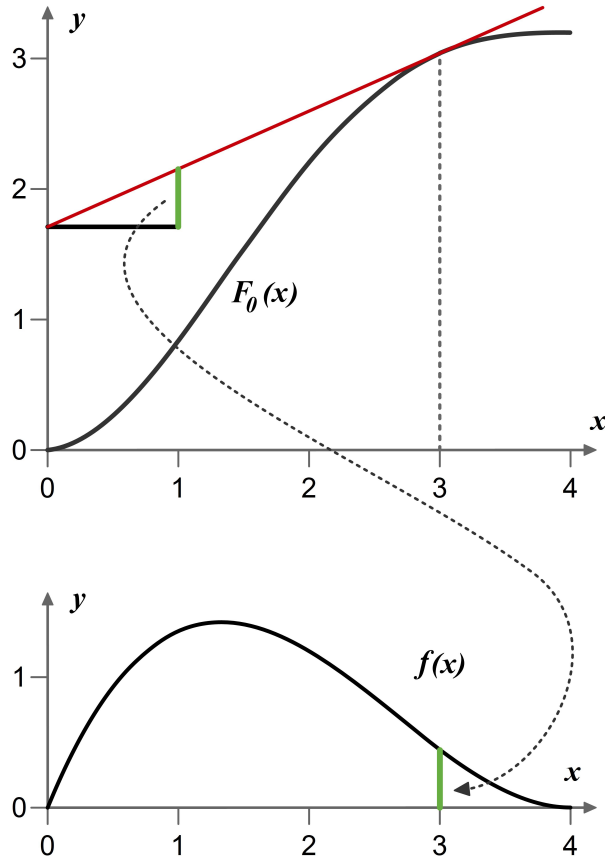


Fig. 4 Slope of the area function at $x = 3$

Why is that?

Let's first look at how the function values of the original function $f(x)$ are related to the slope of the area function $F_0(x)$.

In the next diagram, we can see that the area function $F_0(x)$ rises more steeply in the red region than in the orange region. That happens because the function values of the original function $f(x)$ are larger in the red region, and therefore, between $x = 1$ and $x = 2$, more area is added than in the orange region between $x = 3$ and $x = 4$, where the function values of $f(x)$ are smaller.

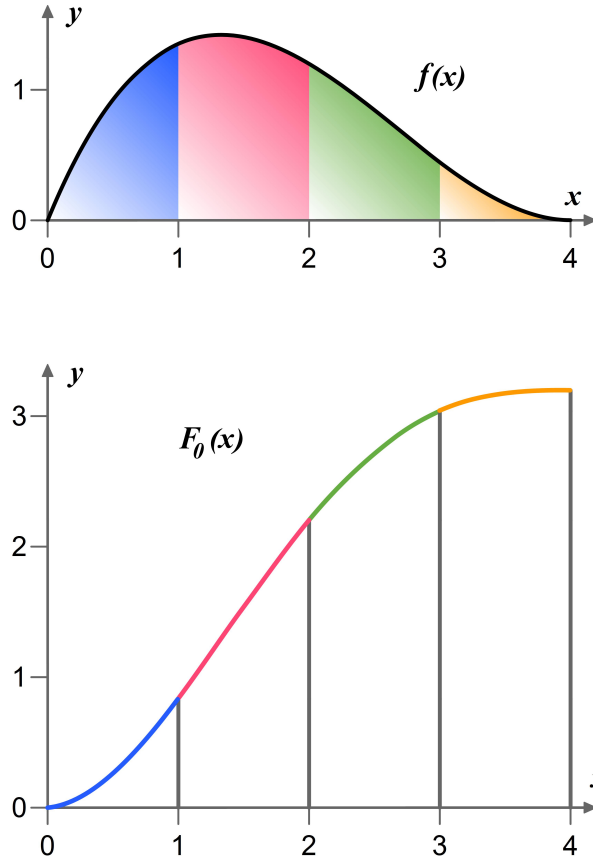


Fig. 5 Function values of $f(x)$ and slopes of $F_0(x)$

So we can see: The greater the slope of the area function $F_0(x)$, the larger the function values of $f(x)$ must be.

We could stop here and be proud that we have reduced the great Fundamental Theorem of Calculus to a nearly vulgar simple idea. But let's go a bit further and get a clear visual sense of why the slope of the area function $F_0(x)$ at every point must be exactly the same as the function value of $f(x)$.

To see this very precisely, we define a piecewise constant function $k(x)$ with the property that the area of the blue regions under the graph of $f(x)$ is exactly equal to the area of the blue region under the graph of $k(x)$. The same should hold for the other colored regions as well.

Because the function $k(x)$ is constant on the interval $(0,1)$, the area function $K_0(x)$ has a constant slope on that interval. The same is true for all other intervals on which $k(x)$ is constant.

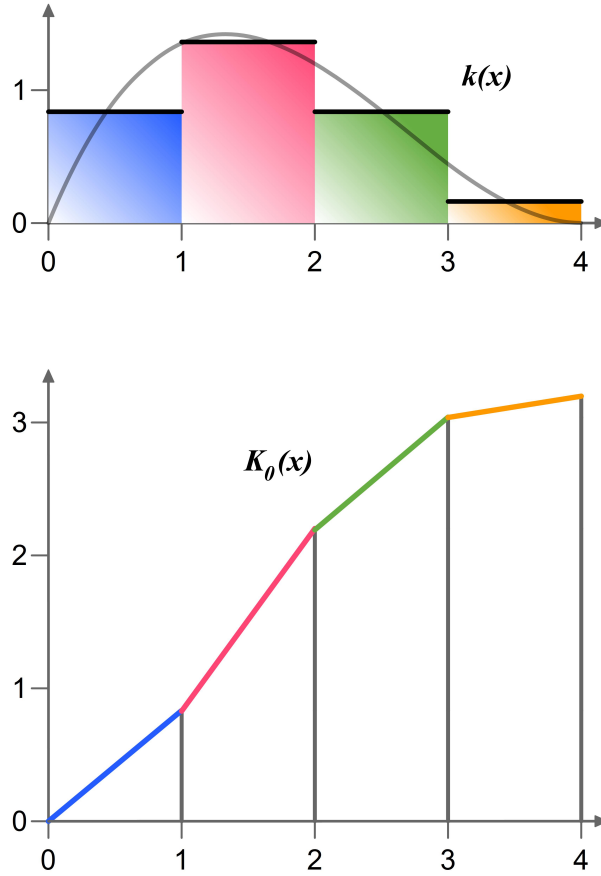


Fig. 6 Function and area function

Now comes the crucial idea: The area of the blue rectangle under the graph of $k(x)$ is calculated using *height times width*. Because the width equals 1, the height of the rectangle is exactly equal to the value of the area. In this case, it is 0.8375. The function value of the area function $K_0(x)$ at $x = 1$ equals the area of the blue rectangle — that is, 0.8375. The slope of the area function $K_0(x)$ on the interval $(0, 1)$ is: *vertical blue segment v divided by horizontal black segment h* , that is, $0.8375 \div 1$ and thus equal to 0.8375 — exactly the same as the function value of $k(x)$ on this interval!

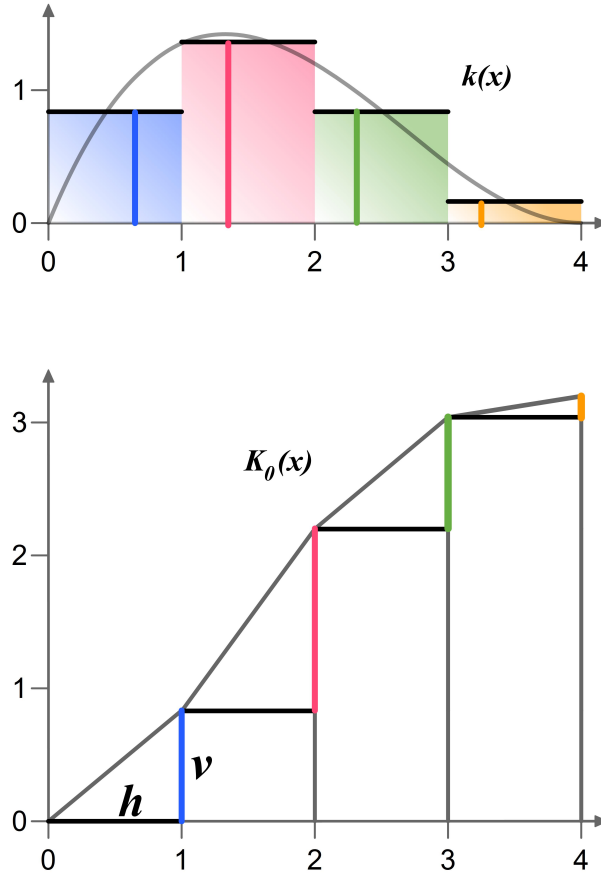


Fig. 7 Piecewise constant function $k(x)$ and piecewise linear area function $K_0(x)$

The same holds true for the other color-marked intervals.

Now, even though we haven't given a fully general intuitive proof of the theorem, we have found a very simple idea that shows clearly why the function values of the original function must be equal to the slopes of the area function — at least for a piecewise constant function $k(x)$ that is similar to $f(x)$.

If we extend our reasoning to all possible piecewise constant functions, we even get an intuitive explanation of the theorem for functions that can get arbitrarily close to the original function $f(x)$.

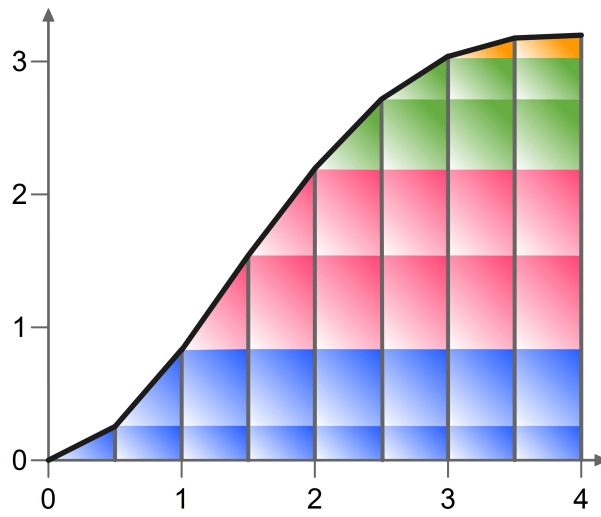
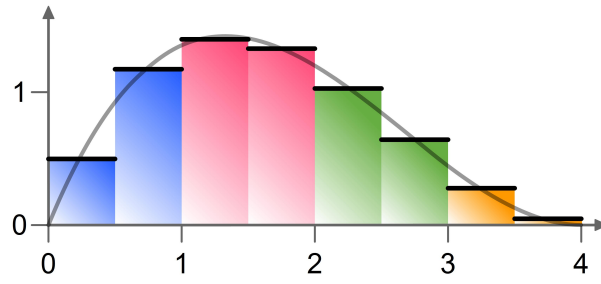


Fig. 8 Finer partition