Multiplying Fractions

In mathematics, the following rule has become standard: Fractions are multiplied by multiplying the numerators and the denominators.
This rule can be summarized briefly as:

> Numerator times numerator,
> denominator times denominator.

Using variables, we can write this very concisely as:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Here, a, b, c, and d stand for natural numbers, with b and d assumed to be nonzero.

Examples

$$\frac{1}{2} \times \frac{1}{3} = \frac{1 \times 1}{2 \times 3} = \frac{1}{6}$$

The numbers being multiplied are called **factors**. In this case, the factors $\frac{1}{2}$ and $\frac{1}{3}$ are being multiplied.

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

The result of a multiplication is called the **product**. Here, the product is the fraction $\frac{8}{15}$.

$$\frac{4}{1} \times \frac{6}{1} = \frac{4 \times 6}{1 \times 1} = \frac{24}{1} = 24$$

When we apply fraction multiplication to natural numbers, we get the results we are familiar with from whole-number multiplication, because $\frac{4}{1}$ equals 4, $\frac{6}{1}$ equals 6, $\frac{24}{1}$ equals 24, and 4×6 equals 24.

To avoid working with unnecessarily large numbers, we simplify fractions before multiplying them.

$$\frac{15}{9} \times \frac{14}{21} = \frac{15 \div 3}{9 \div 3} \times \frac{14 \div 7}{21 \div 7} = \frac{5}{3} \times \frac{2}{3} = \frac{5 \times 2}{3 \times 3} = \frac{10}{9}$$

If there's another opportunity to simplify, we take it.

$$\frac{6}{3} \times \frac{4}{8} = \frac{6 \div 3}{3 \div 3} \times \frac{12 \div 4}{8 \div 4} = \frac{2}{1} \times \frac{3}{2} = \frac{2 \times 3}{1 \times 2} = \frac{\cancel{2} \times 3}{1 \times \cancel{2}} = \frac{3}{1} = 3$$

Sometimes we write a number as a product of prime numbers to make simplification easier.

$$\frac{14}{21} \times \frac{25}{20} = \frac{14 \div 7}{21 \div 7} \times \frac{25 \div 5}{20 \div 5} = \frac{2}{3} \times \frac{5}{4} = \frac{2 \times 5}{3 \times 4} = \frac{2 \times 5}{3 \times 2 \times 2} = \frac{\cancel{2} \times 5}{3 \times \cancel{2} \times 2} = \frac{5}{3 \times 2} = \frac{5}{6}$$

Especially when working with "larger" numbers, it can be helpful to write both numerators and denominators as products of prime numbers right from the start.

$$\frac{14}{21} \times \frac{25}{20} = \frac{2 \times 7}{3 \times 7} \times \frac{5 \times 5}{2 \times 2 \times 5} = \frac{\cancel{2} \times \cancel{7}}{3 \times \cancel{7}} \times \frac{5 \times \cancel{5}}{\cancel{2} \times 2 \times \cancel{5}} = \frac{5}{3 \times 2} = \frac{5}{6}$$

$$\frac{198}{140} \times \frac{60}{99} = \frac{2 \times 3 \times 3 \times 11}{2 \times 2 \times 5 \times 7} \times \frac{2 \times 2 \times 5 \times 3}{3 \times 3 \times 11} = \frac{\cancel{2} \times \cancel{3} \times \cancel{3} \times \cancel{11}}{\cancel{2} \times \cancel{2} \times \cancel{5} \times 7} \times \frac{\cancel{2} \times 2 \times \cancel{5} \times 3}{\cancel{3} \times \cancel{3} \times \cancel{11}} = \frac{6}{7}$$

Phrasing

$\frac{1}{3} \times \frac{1}{4}$ is read as "one third times one fourth" or "one third of one fourth."

We do not say "one third of a fourth" or "one third of the fourth" because we are not referring to a particular fourth, but to the number $\frac{1}{4}$.

In doing so, we transfer the phrasing that we know from whole numbers to the multiplication of fractions, because:

3×4 is "three times four," not "three times a four," or "three times the four," or "three times one of the fours." Similarly, we describe

$\frac{2}{5} \times \frac{9}{7}$ as "two fifths times nine sevenths" or "two fifths of nine sevenths."

We do not say "two fifths of nine times one seventh."

Explaining Fraction Multiplication

We know multiplication of whole numbers as a shortcut for repeated addition, e.g.:

$$3 \times 4 = 4 + 4 + 4 = 12\ .$$

So, 3×4 is three times four.

When we multiply a whole number by a fraction, we want to understand it the same way.

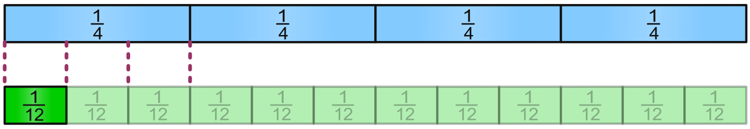
Then, for example, $3 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$. So $3 \times \frac{1}{4}$ is three times one fourth.

We arrive at the same result if we use the definition of fraction multiplication given earlier:

$$3 \times \frac{1}{4} = \frac{3}{1} \times \frac{1}{4} = \frac{3 \times 1}{1 \times 4} = \frac{3}{4}$$

What, then, could $\frac{1}{3} \times \frac{1}{4}$ be?
If $3 \times \frac{1}{4}$ is three times one fourth, then $\frac{1}{3} \times \frac{1}{4}$ could be one third of one fourth.

As we can see, e.g., by looking at the fraction strips, one third of one fourth is one twelfth, because if we divide each fourth into three equal parts, then $3 \times 4 = 12$ of those parts fit into one whole.



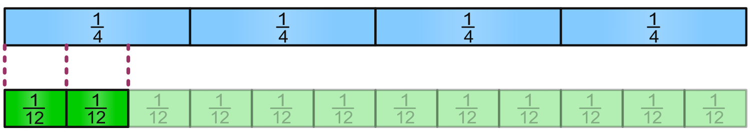
We arrive at the same result if we multiply *numerator times numerator* and *denominator times denominator*.

$$\frac{1}{3} \times \frac{1}{4} = \frac{1 \times 1}{3 \times 4} = \frac{1}{12}$$

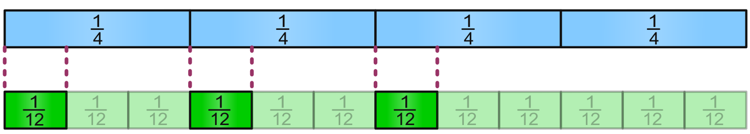
From the natural numbers, we know the following situation: When we calculate, for example, 3×4, we get 12, and when we calculate 6×4, we get *twice* as much as 12, namely 24. Accordingly, the result of $\frac{2}{3} \times \frac{1}{4}$ should be *twice* as large as the result of $\frac{1}{3} \times \frac{1}{4}$. We obtain exactly this result when we use the definition:

$$\frac{2}{3} \times \frac{1}{4} = \frac{2 \times 1}{3 \times 4} = \frac{2}{12}$$

We can also verify this using fraction strips. When we take one third of one fourth, we get one twelfth. When we take *two* thirds of one fourth, we get *two* twelfths.



To illustrate $\frac{1}{3} \times \frac{3}{4}$ with the fraction strips, we could divide $\frac{3}{4}$ into 3 equal parts. That would be correct and reasonable, but then we would not be able to see so clearly that the rule *numerator times numerator and denominator times denominator* is meaningful. Therefore, we choose the following method: We take one third of each of the 3 fourths of the fraction strip. As we can see, that is $\frac{3}{12}$.



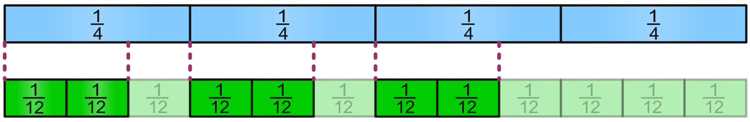
If we multiply according to the definition, exactly the same result occurs:

$$\frac{1}{3} \times \frac{3}{4} = \frac{1 \times 3}{3 \times 4} = \frac{3}{12}$$

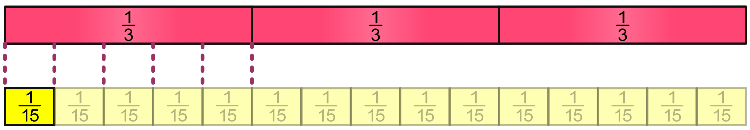
Following this logic, $\frac{2}{3} \times \frac{3}{4}$ should be three times as large as $\frac{2}{3} \times \frac{1}{4}$.
We get this result by using the definition:

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$$

since six twelfths are three times as large as two twelfths.



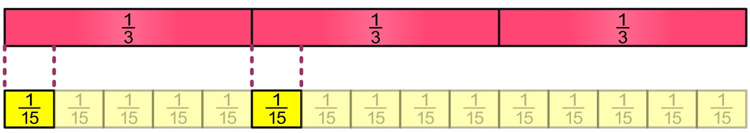
Let us examine fraction multiplication with another example:
If we divide one third into 5 equal parts, each of these parts is one fifteenth, because $5 \times 3 = 15$ of these parts fit into one whole.



The result of $\frac{1}{5} \times \frac{1}{3}$ answers the question: "How much is $\frac{1}{5}$ of $\frac{1}{3}$?"
We calculate:

$$\frac{1}{5} \times \frac{1}{3} = \frac{1 \times 1}{5 \times 3} = \frac{1}{15}$$

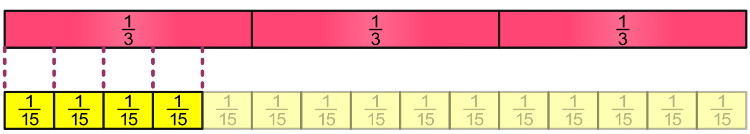
The result of $\frac{1}{5} \times \frac{2}{3}$ answers the question: "How much is $\frac{1}{5}$ of $\frac{2}{3}$?"



We calculate:

$$\frac{1}{5} \times \frac{2}{3} = \frac{1 \times 2}{5 \times 3} = \frac{2}{15}$$

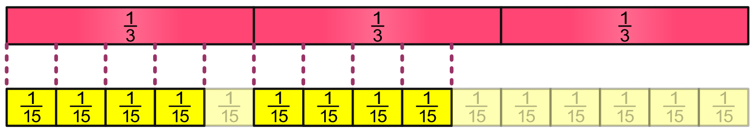
The result of $\frac{4}{5} \times \frac{1}{3}$ answers the question: "How much is $\frac{4}{5}$ of $\frac{1}{3}$?"



We calculate:

$$\frac{4}{5}\times\frac{1}{3}=\frac{4\times 1}{5\times 3}=\frac{4}{15}$$

The result of $\frac{4}{5}\times\frac{2}{3}$ answers the question: “How much is $\frac{4}{5}$ of $\frac{2}{3}$?”



We calculate:

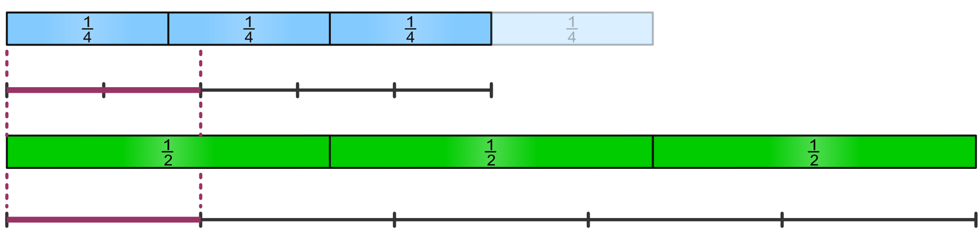
$$\frac{4}{5}\times\frac{2}{3}=\frac{4\times 2}{5\times 3}=\frac{8}{15}$$

Cross-Canceling

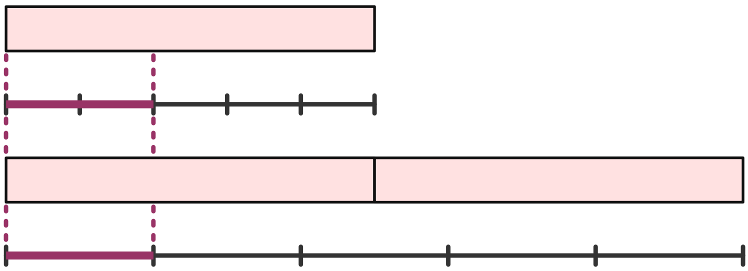
When we want to find $\frac{2}{5}\times\frac{3}{4}$, we simplify before we multiply. So:

$$\frac{2}{5}\times\frac{3}{4}=\frac{2}{5}\times\frac{3}{2\times 2}=\frac{\cancel{2}}{5}\times\frac{3}{\cancel{2}\times 2}=\frac{1}{5}\times\frac{3}{2}$$

But why does this work? Why is it allowed to simplify “across” like that?
Let’s examine this relationship using fraction strips: as we can see, $\frac{3}{2}$ is twice as large as $\frac{3}{4}$.
So instead of taking *two* fifths of $\frac{3}{4}$, we only take *one* fifth of a number that is twice as large as $\frac{3}{4}$, namely $\frac{3}{2}$.



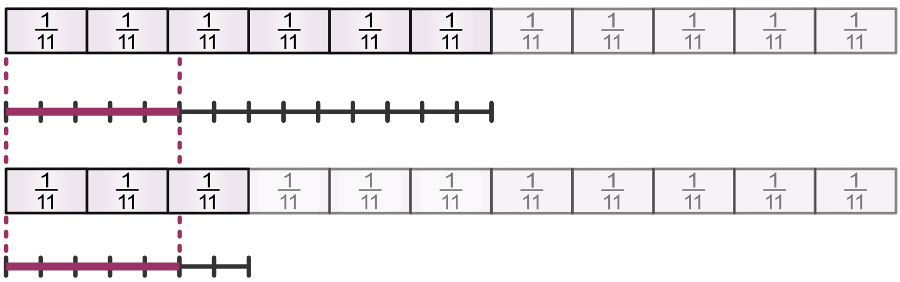
We can also understand this in a more general way: Suppose we are given some length, and we want to take $\frac{2}{5}$ of it. Then we could just as well take $\frac{1}{5}$ of double that length, and we would get the same result.



More Examples

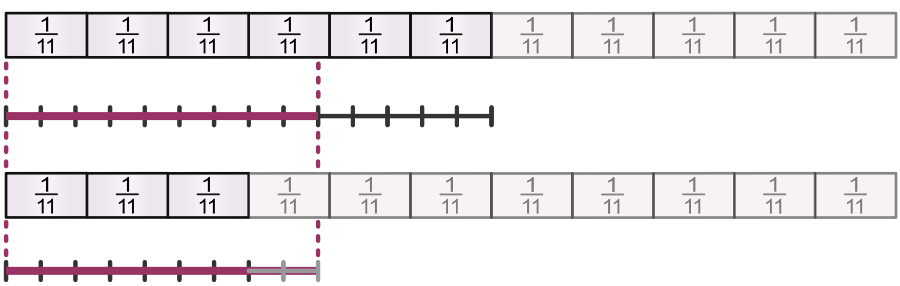
$$\frac{5}{14} \times \frac{6}{11} = \frac{5}{7} \times \frac{3}{11}$$

We divide one bar into 14 parts. And we divide a bar that is half as long into 7 parts. As we can see, the sevenths of the shorter bar are the same length as the fourteenths of the full-length bar.



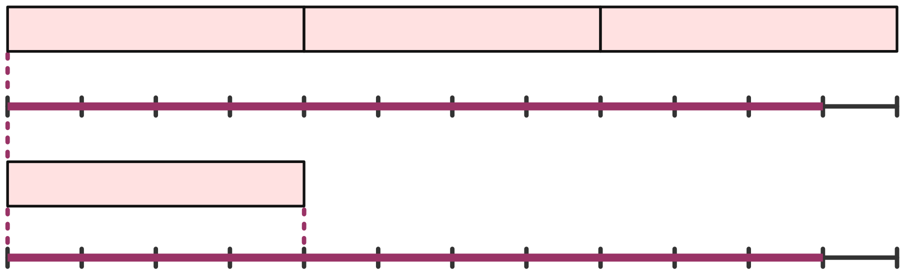
This kind of representation even works when simplifying makes the numerator of the first fraction greater than its denominator.

$$\frac{9}{14} \times \frac{6}{11} = \frac{9}{7} \times \frac{3}{11}$$



In this example, we are working with a segment of arbitrary length a. The twelfths of three times a are the same length as the fourths of the unit-length segment a.

$$\frac{11}{12} \times 3a = \frac{11}{4}a$$



Swapping Numerators and Denominators

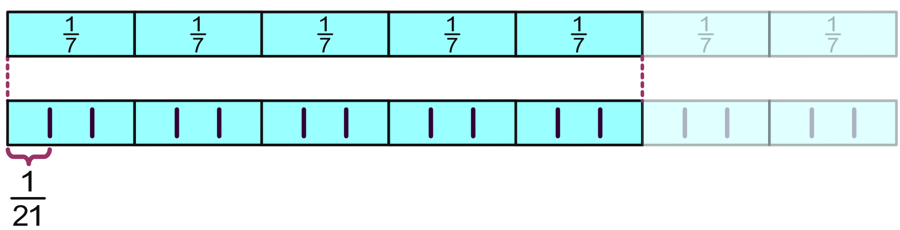
When we multiply two fractions, we can swap the numerators without changing the result. The same applies to the denominators. For example:

$$\frac{2}{3} \times \frac{5}{7} = \frac{5}{3} \times \frac{2}{7} = \frac{5}{7} \times \frac{2}{3} = \frac{2}{7} \times \frac{5}{3}$$

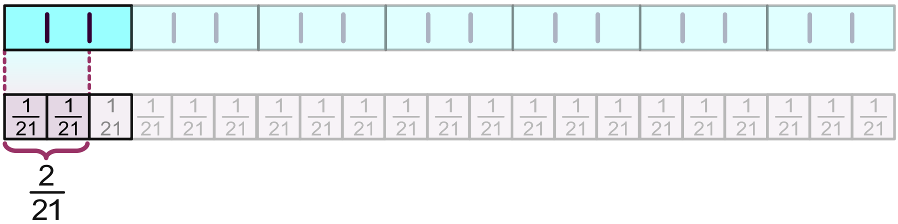
Why does it work?

Original Fractions

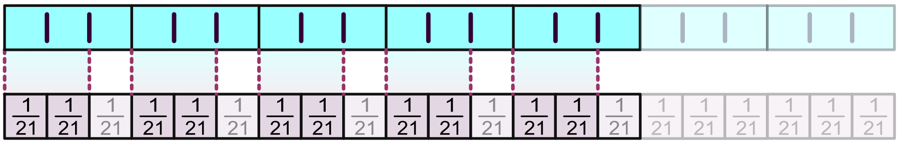
Let's take a look at how we multiply $\frac{2}{3} \times \frac{5}{7}$:
First, we divide all 7 sevenths into 3 equal units. This gives us $3 \times 7 = 21$ units in total. So one of these units is $\frac{1}{21}$ of the whole.



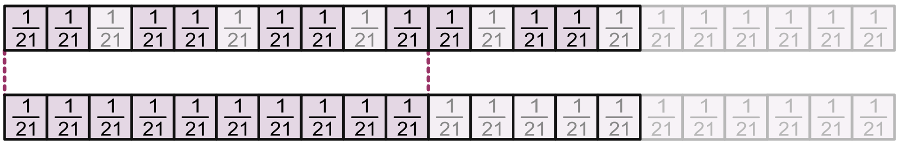
$\frac{2}{3}$ of $\frac{1}{7}$ are *two* units of one part of the fraction, i.e., $\frac{2}{21}$.



$\frac{2}{3}$ of $\frac{5}{7}$ are *two* units of *five* parts, so we get $2 \times 5 = 10$ units in total.

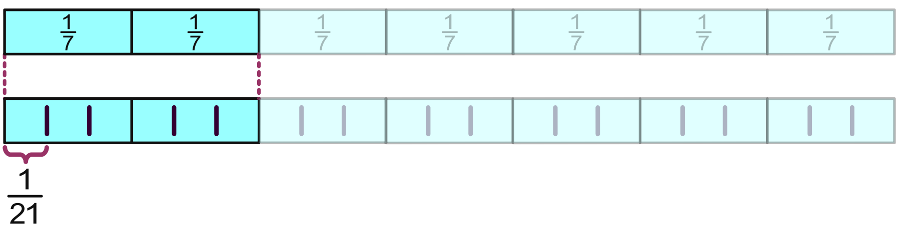


The result is $\frac{10}{21}$.

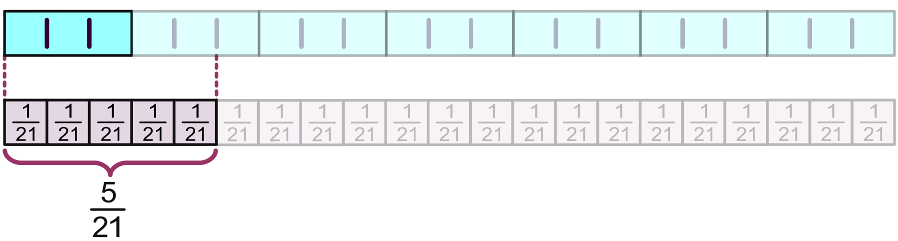


Swapped Numerators

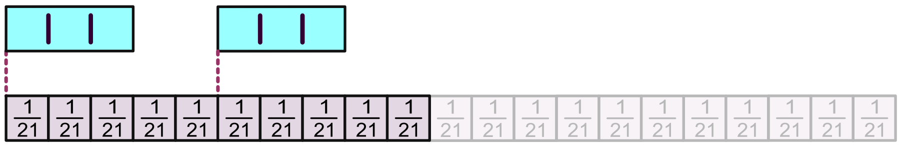
Let's now look at $\frac{5}{3} \times \frac{2}{7}$:
First, we divide all 7 sevenths into 3 units. This gives $3 \times 7 = 21$ units in total.
So each unit represents $\frac{1}{21}$ of the whole.



$\frac{5}{3}$ of $\frac{1}{7}$ are *five* units of *one* part of the fraction, so this gives $\frac{5}{21}$.



$\frac{5}{3}$ of $\frac{2}{7}$ are *five* units of *two* parts, which gives $5 \times 2 = 10$ units in total.

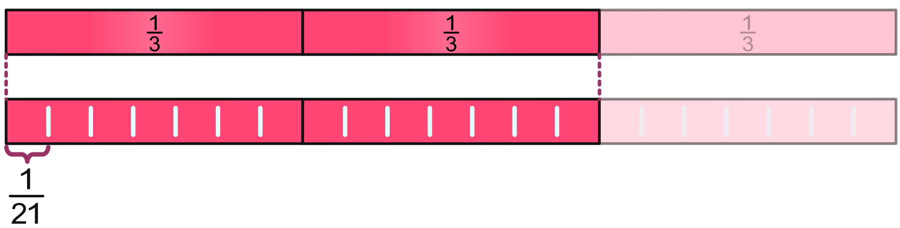


So the result is $\frac{10}{21}$.

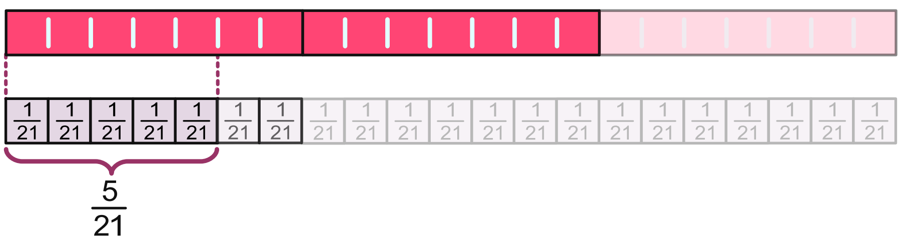
Why does the result stay the same?
Since the denominators stayed the same, the units stayed the same too — they are twenty-firsts.
The number of units in the result also stayed the same: In the first case, we took *two* units of *five* parts; in the second case, we took *five* units of *two* parts. In both cases, the result is *10 units*.

Swapping Denominators

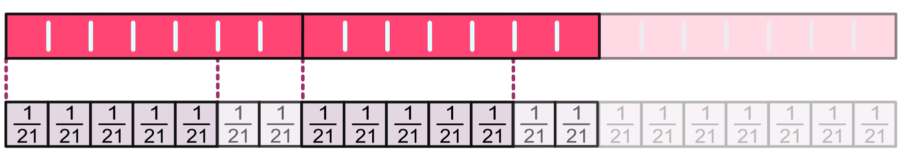
Let's now look at $\frac{5}{7} \times \frac{2}{3}$:
First, we divide each of the 3 thirds into 7 units. This gives us a total of $7 \times 3 = 21$ units.
Each unit is therefore $\frac{1}{21}$ of the whole.



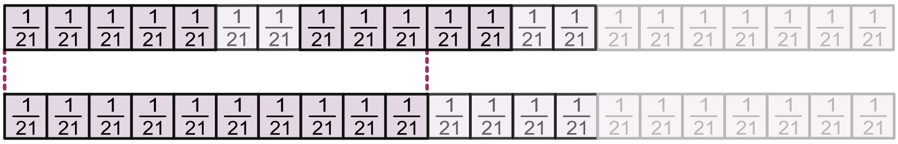
$\frac{5}{7}$ of $\frac{1}{3}$ are *five* units of *one* part of the fraction, so the result is $\frac{5}{21}$.



$\frac{5}{7}$ of $\frac{2}{3}$ means *five* units of *two* parts, which gives us a total of $5 \times 2 = 10$ units.



So the result is $\frac{10}{21}$.

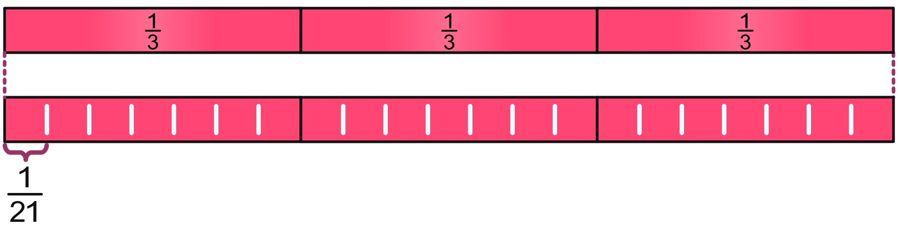


Why does $\frac{5}{7} \times \frac{2}{3}$ give the same result as $\frac{5}{3} \times \frac{2}{7}$?
Since the numerators remain the same, we are again taking *five* units of *two* parts, which gives us 10 units.

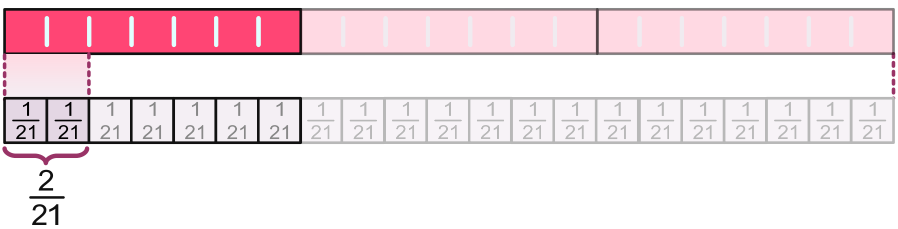
In the previous calculation, we divided 7 sevenths into 3 units each, resulting in 21-sths. In this calculation, we divide 3 thirds into 7 units each— again resulting in 21-sths. That's why the denominator stays the same.

Swapped Numerators (Again)

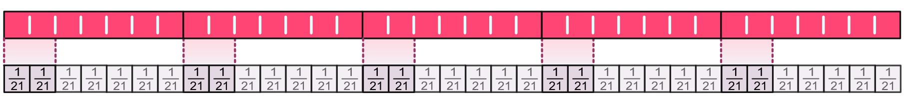
Finally, we have $\frac{2}{7} \times \frac{5}{3}$:
First, we divide each of the 3 thirds into 7 units. This gives us $7 \cdot 3 = 21$ units in total. So one unit is $\frac{1}{21}$ of the whole.



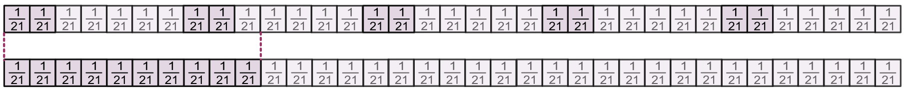
$\frac{2}{7}$ of $\frac{1}{3}$ are *two* units of *one* part of the fraction, which gives us $\frac{2}{21}$.



$\frac{2}{7}$ of $\frac{5}{3}$ means we take *two* units of *five* parts, which gives us $2 \times 5 = 10$ units.



So the result is $\frac{10}{21}$.



Why do we get the same result from $\frac{2}{7} \times \frac{5}{3}$ as from all the previous products?
The numerator stays the same because we are either taking *five* units of *two* parts or *two* units of *five* parts. The total number of units is the same in both cases.
The denominator stays the same because we are either dividing 3 thirds into 7 units each or dividing 7 sevenths into 3 units each. The total number of units is the same in both cases.

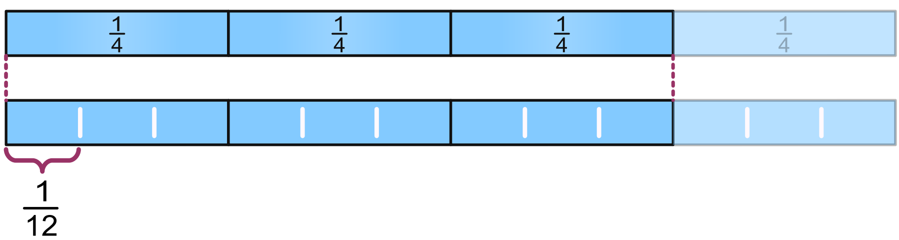
Examples

Let's take a look at a few more examples:

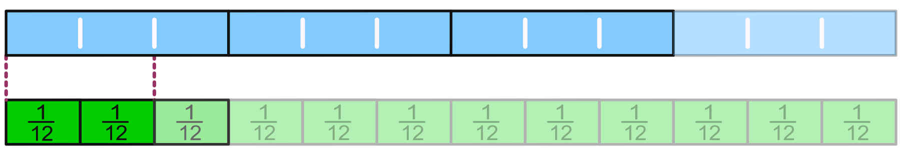
Example 1

We want to multiply the fractions $\frac{2}{3}$ and $\frac{3}{4}$, that is, we want to calculate $\frac{2}{3} \times \frac{3}{4}$.
We ask ourselves: What are two thirds of three fourths?

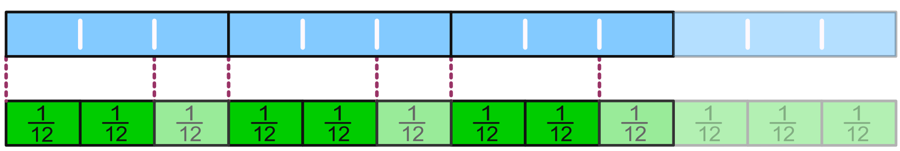
First, we divide each of the 4 fourths into 3 units. This gives us $3 \times 4 = 12$ units. So one unit is $\frac{1}{12}$ of the whole.



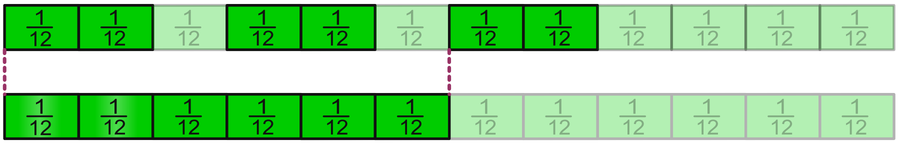
$\frac{2}{3}$ of $\frac{1}{4}$ are 2 units.



$\frac{2}{3}$ of $\frac{3}{4}$ are then $2 \times 3 = 6$ units.

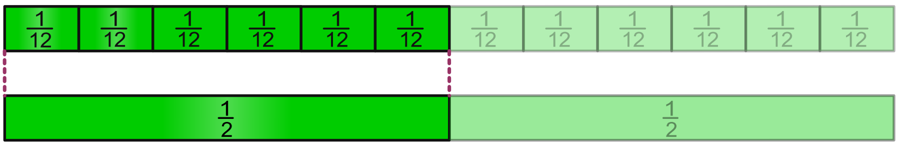


Put together, these are $\frac{6}{12}$.



Because we simplify the result, we now get the following calculation:

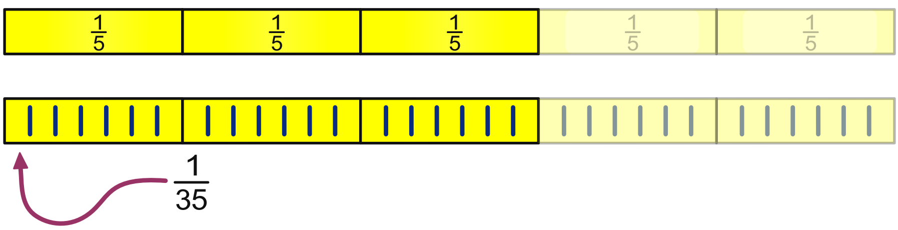
$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2}$$



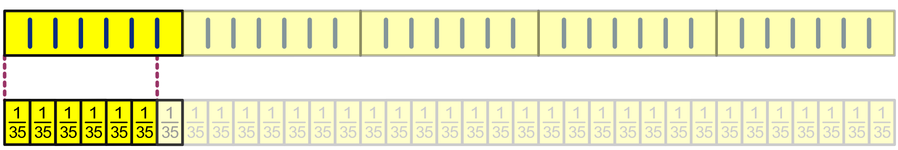
Example 2

We want to multiply the fractions $\frac{6}{7}$ and $\frac{3}{5}$, so we compute $\frac{6}{7} \times \frac{3}{5}$.
We ask ourselves: What are six sevenths of three fifths?

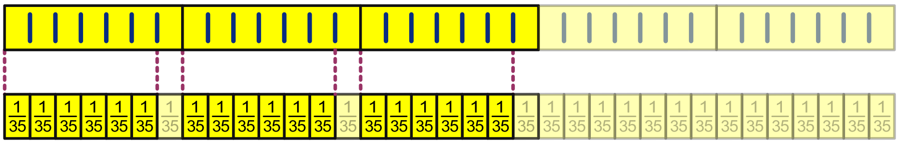
First, we divide all 5 fifths into 7 units. That gives us $7 \times 5 = 35$ units in total. So one unit is equal to $\frac{1}{35}$ of the whole.



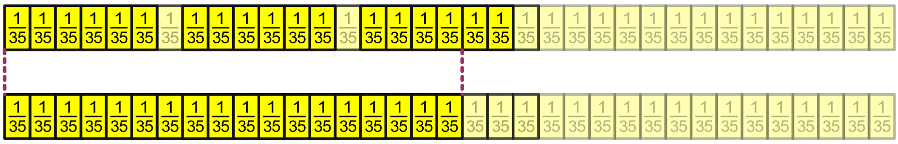
$\frac{6}{7}$ of $\frac{1}{5}$ are 6 units.



$\frac{6}{7}$ of $\frac{3}{5}$ means we take *6* units of *3* parts. That gives us $6 \times 3 = 18$ units.



Put together, that gives us $\frac{\mathbf{18}}{\mathbf{35}}$.



Here is the full calculation:

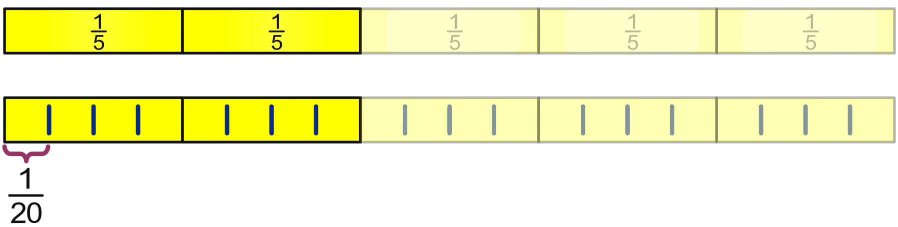
$$\frac{6}{7} \times \frac{3}{5} = \frac{6 \times 3}{7 \times 5} = \frac{18}{35}$$

Example 3

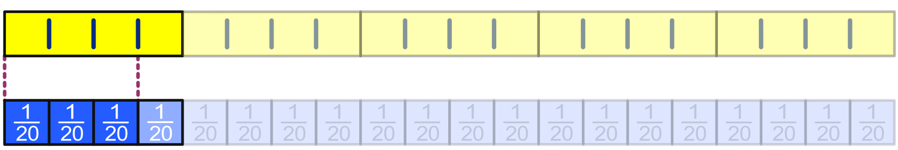
We want to multiply the fractions $\frac{\mathbf{3}}{\mathbf{4}}$ and $\frac{\mathbf{2}}{\mathbf{5}}$, so we compute $\frac{\mathbf{3}}{\mathbf{4}} \times \frac{\mathbf{2}}{\mathbf{5}}$.

We ask ourselves: What are three fourths of two fifths?

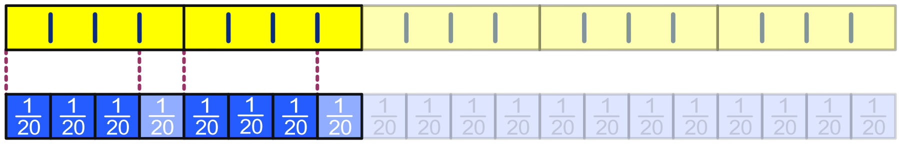
First, we divide all 5 fifths into 4 units. That gives us $4 \times 5 = 20$ units. So one unit is $\frac{\mathbf{1}}{\mathbf{20}}$ of the whole.



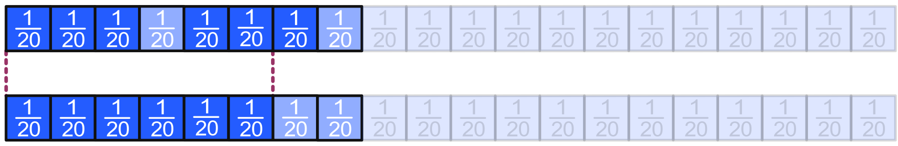
$\frac{\mathbf{3}}{\mathbf{4}}$ of $\frac{\mathbf{1}}{\mathbf{5}}$ are 3 units.



$\frac{\mathbf{3}}{\mathbf{4}}$ of $\frac{\mathbf{2}}{\mathbf{5}}$ are *3* units of *2* parts. That gives us $3 \times 2 = 6$ units.

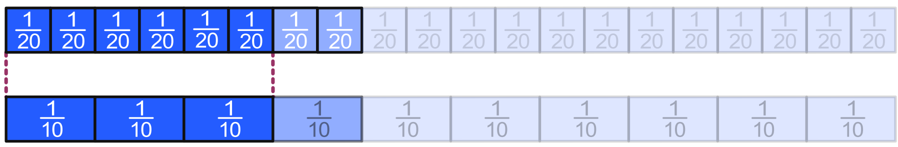


Put together, that gives us $\frac{6}{20}$.



Since we simplify the result, we now get the following calculation:

$$\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20} = \frac{6 \div 2}{20 \div 2} = \frac{3}{10}$$

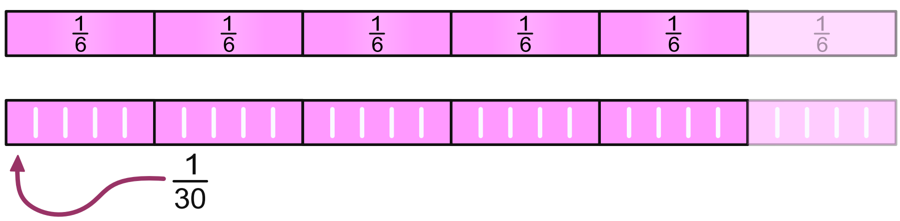


Example 4

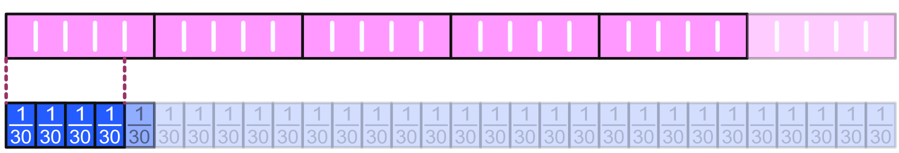
We want to multiply the fractions $\frac{4}{5}$ and $\frac{5}{6}$, so we compute $\frac{4}{5} \times \frac{5}{6}$.

Let's ask ourselves: What are four fifths of five sixths?

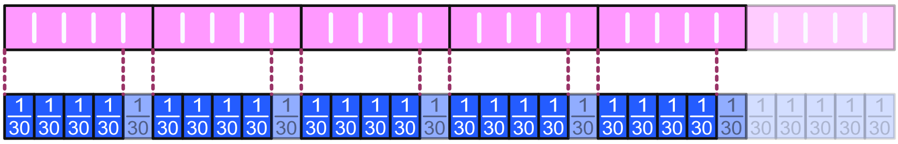
First, we divide all 6 sixths into 5 units. That gives us $5 \times 6 = 30$ units. So one unit is $\frac{1}{30}$ of the whole.



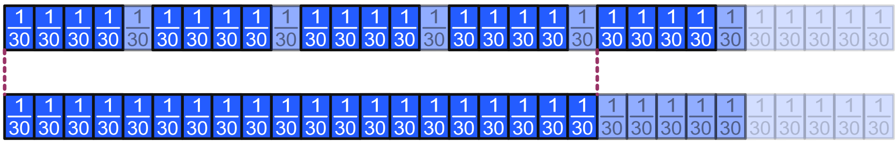
$\frac{4}{5}$ of $\frac{1}{6}$ are 4 units.



$\frac{4}{5}$ of $\frac{5}{6}$ are *4* units of *5* parts. That gives us $4 \times 5 = 20$ units.

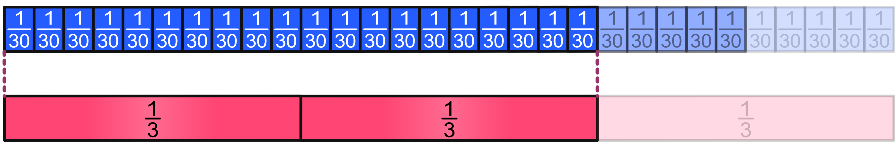


Put together, that gives us $\frac{20}{30}$.



Since we simplify the result, we now get the following calculation:

$$\frac{4}{5} \times \frac{5}{6} = \frac{4 \times 5}{5 \times 6} = \frac{20}{30} = \frac{20 \div 10}{30 \div 10} = \frac{2}{3}$$



By the way, we would have gotten the same result if we had simplified the fractions *before* multiplying. But then, there would have been nothing left to multiply.

$$\frac{4}{5} \times \frac{5}{6} = \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{6} = \frac{4}{1} \times \frac{1}{6} = \frac{2 \times 2}{1} \times \frac{1}{2 \times 3} = \frac{\cancel{2} \times 2}{1} \times \frac{1}{\cancel{2} \times 3} = \frac{2}{1} \times \frac{1}{3} = \frac{2}{3}$$