Understanding Equivalence Transformations

In this section, one way of visualizing equivalence transformations is shown, which certainly will not make sense to everyone. And precisely for that reason, it is included here! Different people understand mathematics differently. In many respects, it is beneficial to immerse oneself in the mathematical understanding of others. Ideally, this should repeatedly take place in learning groups. Showing your own thoughts to others builds confidence and pride. Moreover, an individual perspective on mathematics can never be wrong.

In school mathematics, an equivalence transformation modifies an equation so that the transformed equation has the same solution set as the original equation. There are mainly 5 types of transformations: On both sides of the equation, the same

1) can be added, or
2) subtracted, or both sides of the equation can be multiplied by the same nonzero number,
3) multiplied, or
4) divided. The final possibility is
5) an expression manipulation on just one side of the equation.

How can one visualize that an equivalence transformation does not change the solution set?

When equivalence transformations are explained, this usually happens with a balance scale. Therefore, we want to take a different approach here and look at how equations with their equivalence transformations appear on the number line. For this, we consider the equation

$$3x - 4 = x + 2$$

together with its solution set $\mathbb{L} = \{3\}$.

If we substitute $x = 3$, this equation looks on the number line as follows:

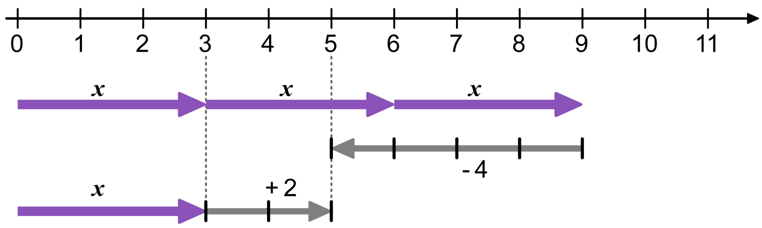

Fig. 1 $3x - 4 = x + 2$

That the solution set of the equation $3x - 4 = x + 2$ equals $\mathbb{L} = \{3\}$ means not only that this equation is correct for $x = 3$, but also that it is false for all other numbers we could substitute for x. We can also visualize this on the number line. Let's try $x = 3.5$.

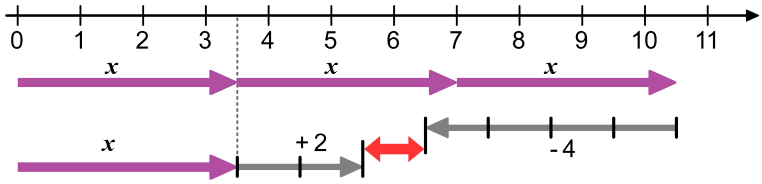

Fig. 2 $3x - 4 > x + 2$

As we can see, the equation is now false because the left side is greater than the right side. Even if we substitute larger numbers for x, nothing changes — something we can reason about without further graphics.
If we substitute a number smaller than 3 for x — for example, 2.5 — the equation is also false because then the right side is greater than the left side. And this situation does not change even if we substitute even smaller numbers.

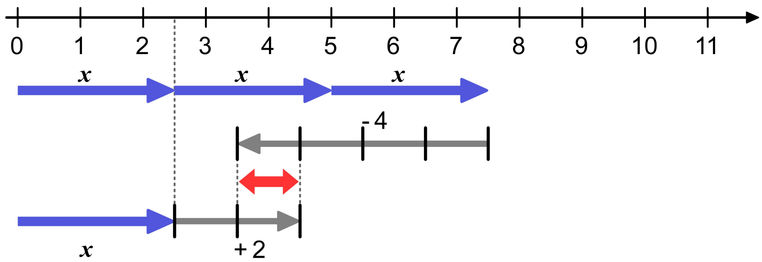

Fig. 3 $3x - 4 < x + 2$

We have now visually justified why the number 3 is indeed the only solution of the equation $3x - 4 = x + 2$. But does this also hold if we apply equivalence transformations to this equation, for example, if we add or subtract x on both sides of the equation? After all, at the beginning of the solution process, we do not know how large x must be.
Let's just try something! What happens if we add an x to both sides in the situation from Fig. 3?

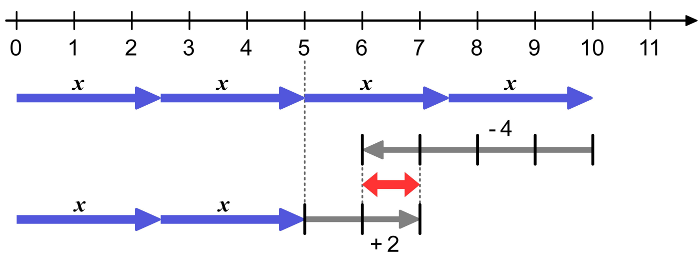

Fig. 4 $4x - 4 < 2x + 2$

Wie wir sehen, hat sich der ”‘interessante”’ Teil der Gleichung auf der Zahlengerade nach rechts verlagert. Nun gut, dann können wir ja auch auf beiden Subtract $2x$ from both sides.

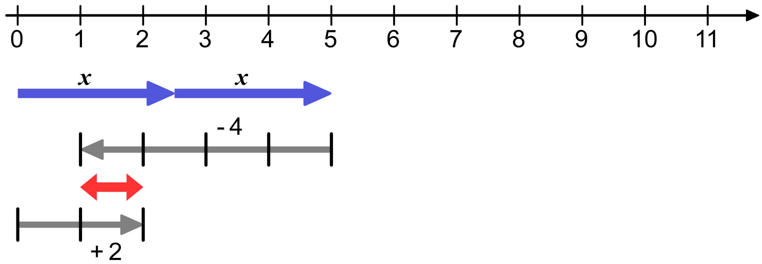

Fig. 5 $2x - 4 < 2$

In all three cases, the amount by which the left side is smaller than the right side remains the same. So we could add or subtract as many x's as we want without changing the difference in size between the two sides of the equation.
Let's look again at the situation in Fig. 2 and add x to both sides. Then — as in Fig. 2 — the left side is larger than the right side by the same amount.

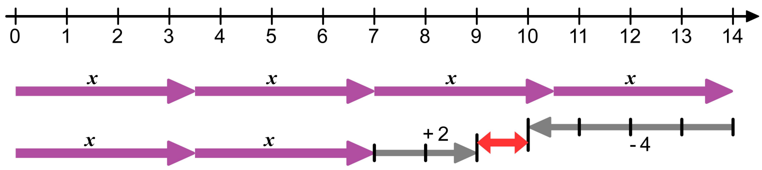

Fig. 6 $4x - 4 > 2x + 2$

The difference also remains the same if we subtract $2x$ from both sides. This means: In this way, we do not change the solution set of the equation, because the "'interesting"' part of the equation is not affected by these equivalence transformations at all.

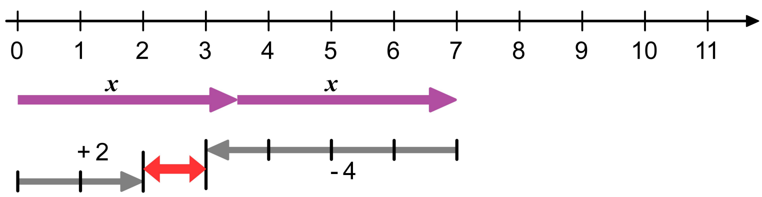

Fig. 7 $2x - 4 > 2$

It is now probably no longer surprising that $x = 3$ remains the correct solution when we transform the original equation $3x - 4 = x + 2$ by subtracting x from both sides.

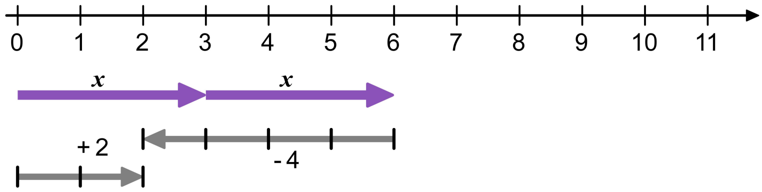

Fig. 8 $2x - 4 = 2$

Encouraged by our findings, we can now even introduce a "‘new"’ equivalence transformation, namely the addition of *something* to both sides of an equation — here abbreviated by *something*.

$$\begin{array}{rrcll} & 2x-4 & = & 2 & \quad \mid + \textit{something} \\ \Leftrightarrow & 2x-4+\textit{something} & = & 2+\textit{something} & \end{array}$$

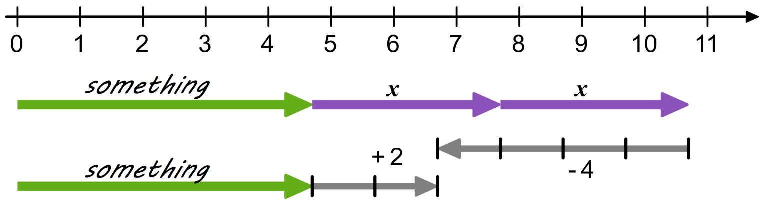

Fig. 9 $3x - 4 +$ *something* $= x + 2 +$ *something*