

Understanding Expression Manipulation

This text shows a few ways one can approach understanding expression manipulation.

Starting in 7th grade, expression manipulation (or algebraic transformations) are on the curriculum. These are procedures used to rewrite expressions. Such procedures are usually taught by showing examples of how expressions are rewritten. Students are then expected to perform these transformations on similar expressions as well.

If one wants to understand expression manipulation, one has to go a bit further back and define what expressions are and also what *equivalent* expressions are.

Definition: Expressions are combinations of numbers, variables, and arithmetic operations that can be evaluated.

To focus on understanding expression manipulation, we assume here that expressions are sufficiently familiar.

Definition: Two expressions are called equivalent if they always yield the same result, no matter what numbers are substituted for the variables. For the same variables, the same numbers should be substituted.

When one performs an expression manipulation in mathematics, it always means rewriting an expression into an *equivalent* expression. A transformation is therefore correct if an expression is rewritten into an *equivalent* expression and it is incorrect if an expression is not rewritten into an equivalent expression.

Let's look at an example: Consider the two expressions

$$5 - 2 + a \text{ and } 5 - (2 - a)$$

If we substitute $a = 3$, we get:

$$5 - 2 + 3 \text{ and } 5 - (2 - 3)$$

In both cases, the result is 6.

If the expression manipulation from the term $5 - 2 + a$ to the term $5 - (2 - a)$ is supposed to be correct, both terms must yield the same result for *all* numbers that can be substituted for a .

Here is the point where *mathematical* understanding begins. First, one could clarify what the claim of equivalence of results between the two terms actually states. There are many ways to approach this, and many levels of understanding. In a sense, one can interpret this claim as follows: Whenever in the future we substitute the same number for the variable a in both terms and then evaluate the terms, we get the same result. Normally, we assume not to know what will happen in the future. In the case of expressions, however, we claim to know it exactly. Why does this work? Or does it perhaps not work at all?

Since there are infinitely many numbers that can be substituted for the variables in an expression, expression manipulations cannot be verified as correct based on experience, because no one could ever test their correctness with infinitely many

numbers. There must therefore be something else fascinating in mathematics that allows us to be convinced of the correctness of an expression manipulation even for numbers we have not yet tried.

If one wants to know whether a statement is correct for *all* numbers, one might first ask whether one can even find numbers for which the statement is correct. So one could calculate a few examples. But what would happen if we substituted numbers into two supposedly equivalent terms and obtained different results? Our first guess would probably be that we miscalculated — raising the next question: Can we always determine whether a calculation was done correctly? Do all calculations in the world always yield only one single result?

These questions, which may seem somewhat trivial, actually lead to non-trivial subfields of mathematics such as recursion theory and complexity theory. We will not go into that here, but it shows how diverse the answers to such simple questions can be.

Returning to our two terms: there is a standard model that allows us to visualize calculations, namely the number line. If we substitute the number 3 for a , we get the following picture with the left term:

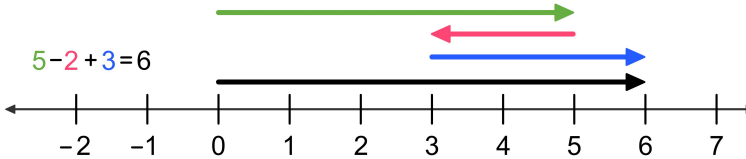


Fig. 1 $5 - 2 + 3 = 6$

To find the result of the right term, we first evaluate the parentheses.

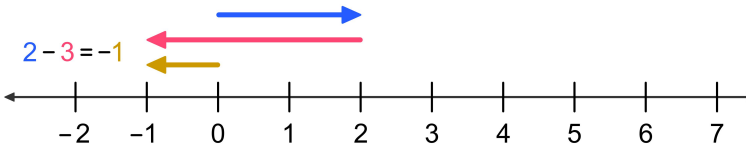


Fig. 2 $2 - 3 = -1$

We then use the result of the parentheses to determine the result of the term. To what extent $-(-1)$ is actually equal to $+1$ will not be discussed here. (See "Why is Minus times Minus Plus?")



Fig. 3 $5 - (-1) = 5 + 1 = 6$

Now we have seen that both terms give the same result when we substitute the

number 3 for the variable a . To see whether this also works with other numbers, the arrow representing the variable a will be lengthened slightly.

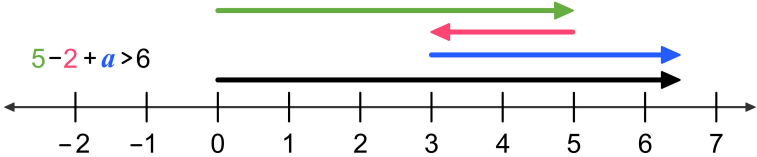


Fig. 4 $5 - 2 + a > 6$

Unsurprisingly, the result of the term is now greater than 6, because we have added a number larger than 3 to $5 - 2$.

On the other hand, it also works. First we evaluate the parentheses and obtain an arrow b , which is longer than the arrow representing -1 .

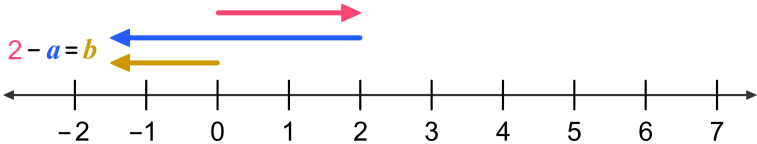


Fig. 5 $2 - a = b$

Previously, we added the number 1 to 5. Now we add a number b , which is greater than 1, and the result is greater than 6.

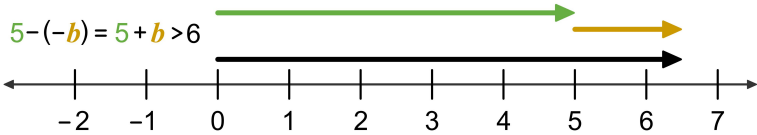
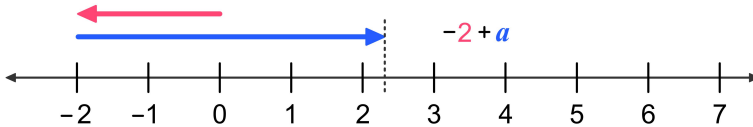
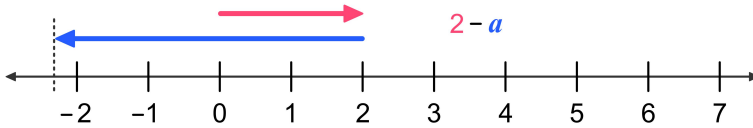


Fig. 6 $5 - (-b) = 5 + b > 6$

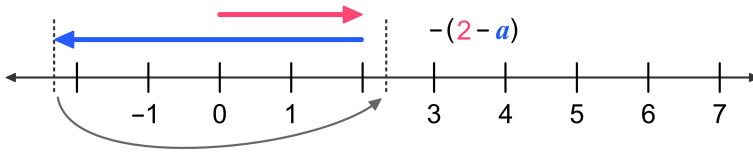
In summary, we can note: If we use a longer blue arrow, the brown arrow also becomes longer. If the brown arrow is longer, the result is also greater. Since we have not specified which exact number we substituted for a and still obtained the same results, this indicates that there is an abstract pattern that makes the two terms expression-equivalent. Recognizing such a pattern is another major step in developing a student's mathematical way of thinking. What could this pattern be? Here is a suggestion for the first steps of such reasoning: First, we no longer focus on the 5, because it appears in the same position in both terms and thus probably has nothing to do with the pattern that might be responsible for the expression equivalence. In the left term we are then left with $-2 + a$, which we can visualize as follows.

Fig. 7 $-2 + a$

In the right term, we first focus on the parentheses.

Fig. 8 $2 - a$

The minus sign in front of the parentheses then flips the intermediate result to the other side.

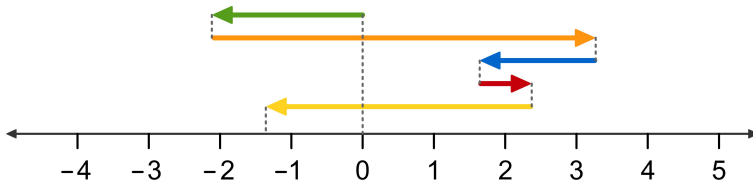
Fig. 9 $-(2 - a)$

This process can also be described in a very mundane way: "If I first go a certain distance to the left and then a certain distance to the right, I arrive at the same point as if I first go to the right, then to the left, and finally flip the result."

The good thing about such an explanation is that no concrete numbers appear in it. Presumably, the two terms yield the same result for quite a few numbers!

To understand the expression equivalence of these terms, it can also be helpful to generalize the abstract pattern found. In this case, a mnemonic rule suggests itself: "A minus sign in front of parentheses flips the signs of all terms inside the parentheses."

If the variables a, b, c, d, e represent positive numbers, we see the situation $-a + b - c + d - e$:

Fig. 10 $-a + b - c + d - e$

In the next picture, we first invert everything inside the parentheses and then

flip the intermediate result to the other side. The final result is then the same as above.

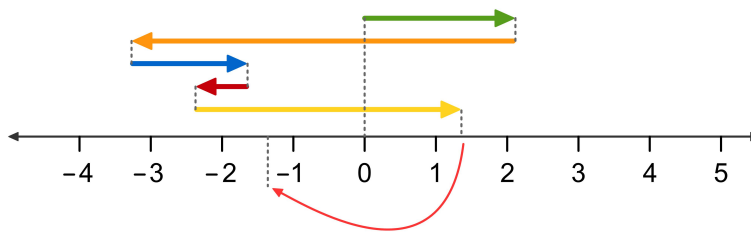


Fig. 11 $-(a - b + c - d + e)$

We have now discovered quite a lot: We left out the 5 because it appears in both terms equally and, visually, has nothing to do with the expression equivalence of the two terms. With a bit more work, we could derive a rule from this, such as: The same number can be subtracted from two terms without affecting whether the terms are expression-equivalent or not.

Furthermore, we have seen how to handle minus signs in front of parentheses, from which we can derive the following rule: If a term is a sum, we can

1. change the signs of all summands,
2. enclose the entire term in parentheses, and
3. place a minus sign in front of the parentheses

without changing the result of the term.

The justification is as simple as the back-and-forth movement we know from everyday life. And in doing so, we gain an additional insight whose value cannot be overestimated: Behind a seemingly complicated "pile of formulas" there can be a very simple idea.

There are many other questions whose answers could contribute to understanding, e.g.:

- Why do we even perform expression manipulations?

Answer: With expressions, for example, we indicate how something is calculated. If we want to show someone how to calculate something, courtesy dictates that the expression should first be transformed into the simplest possible form before passing it on.

- How is this actually done in proper mathematics (i.e., university mathematics)?

Answer: The case considered is a special case of the distributive law, which is not proven but rather required axiomatically for the real numbers. This also shows that mathematics is not true because the axioms are true and the consequences are truth-preserving, but the axioms are constructed to fit what we are already convinced of.

- Can the correctness of this expression manipulation also be demonstrated using written addition and subtraction?

Answer: Yes.

Understanding a mathematical relationship is not a constant state of mind; it arises anew each time, is always slightly different, and is never complete, because ultimately we cannot answer why mathematics – in particular, as in the example of two terms here, and in general – works at all. Or, in other words: If God created the world, why did He make it mathematically?