Infinitely Wide Area with a Finite Surface Area

Using integral calculus, we can calculate the areas enclosed between a function graph and the x-axis within certain limits. Sometimes, however, this also works when, for example, the right limit goes to infinity. Then we get an infinitely wide area that can still have a finite surface area.
But how can that be? Shouldn't an area that is infinitely wide also have an infinitely large surface area?
Our intuition tells us: Even if, for example, a rectangle has only a "small" height, its area will still grow without bound when the rectangle becomes wider and wider.

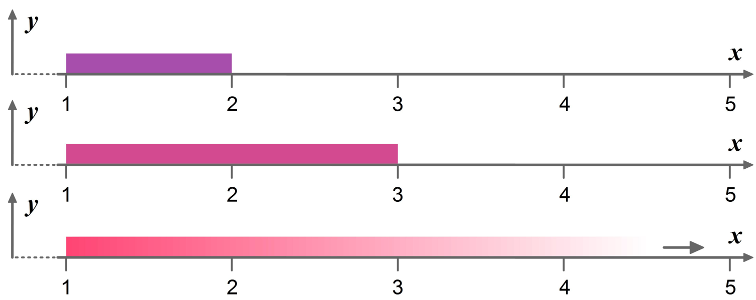

Fig. 1 Infinitely increasing surface area

Let's look at a concrete example: Here we have part of the function graph of $f(x) = 0.5^x$.

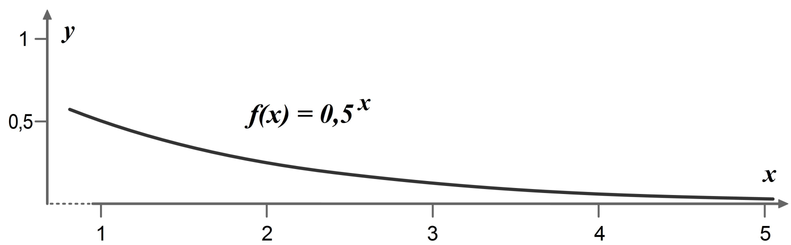

Fig. 2 Graph of the function $f(x) = 0.5^x$

Now we can determine how large the surface area of the blue region is.

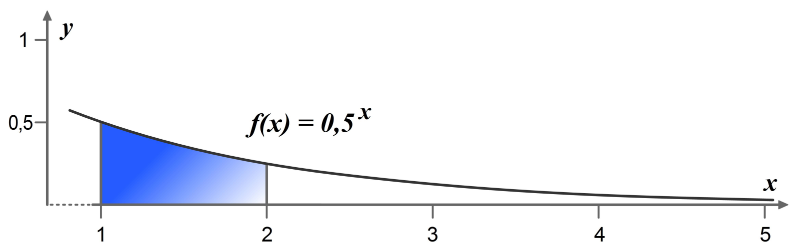

Fig. 3 Area between the graph and the x-axis from 1 to 2

Mathematically, it looks like this:

$$\int_1^2 0.5^x \, dx = \left[-\frac{1}{\ln(2)} \times 0.5^x \right]_1^2 \approx 0.3607$$

So the blue area has a surface area of about 0.3607 square units.
Now we can think about what happens to the surface area when we move the right boundary farther and farther to the right. Then we get an area shown by the green region.

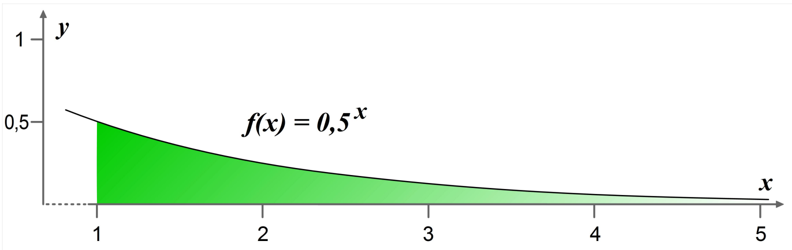

Fig. 4 Area between the x-axis and the graph of $f(x) = 0.5^x$

The farther we move the right boundary to the right, the wider the area becomes and the larger its surface area. Yet the total surface area does not become infinite! It can be difficult to imagine. Maybe that's because of a way of thinking expressed in the saying "the straw that broke the camel's back." Even if what's added each time is very small — like a drop of water — the amount of water still grows without limit if you keep adding drops often enough.
In the case of our function, we can imagine adding to the blue area the region between 2 and 3, then the region between 3 and 4, then between 4 and 5, and so on. What's being added keeps getting smaller, but the total area keeps getting larger. How can something that keeps getting larger infinitely many times *not* become infinitely large?
To understand the principle, let's imagine covering the green infinitely wide area with a single sheet of paper. So we take a square sheet of paper with side length 1 and place it over the interval from 1 to 2.

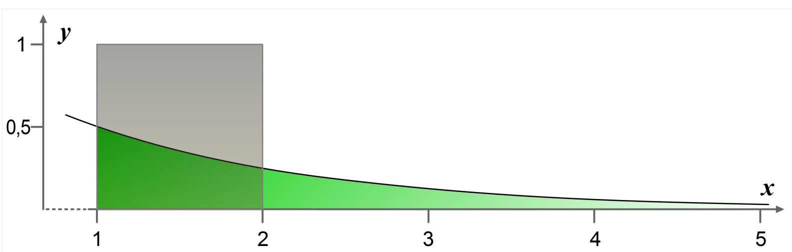

Fig. 5 One square unit

Now we cut the sheet in half ...

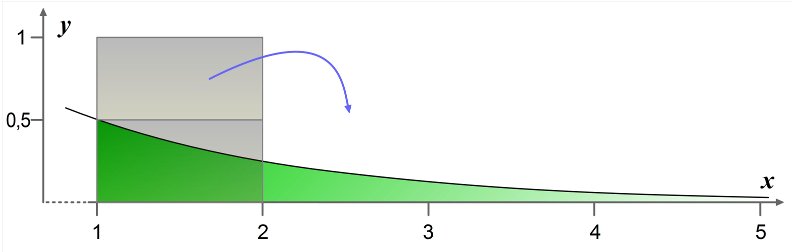

Fig. 6 One square unit cut into two halves on top of each other

... and place the upper half next to the lower half.

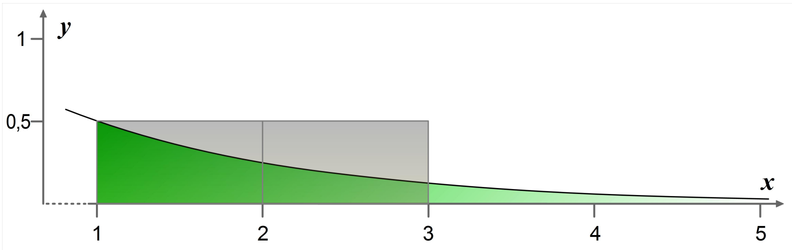

Fig. 7 One square unit cut into two halves side by side

Now we cut the right half in half again ...

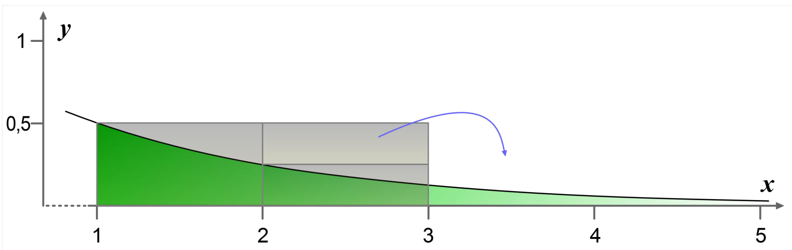

Fig. 8 The second half cut into two halves on top of each other

... and place the upper half to the right of the lower half.

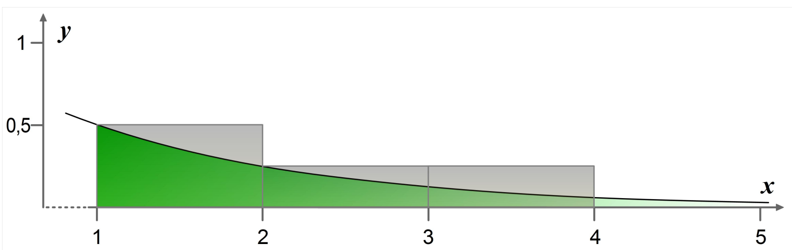

Fig. 9 The second half cut into two halves side by side

We can keep doing this — at least in thought: cut the right sheet in half again ...

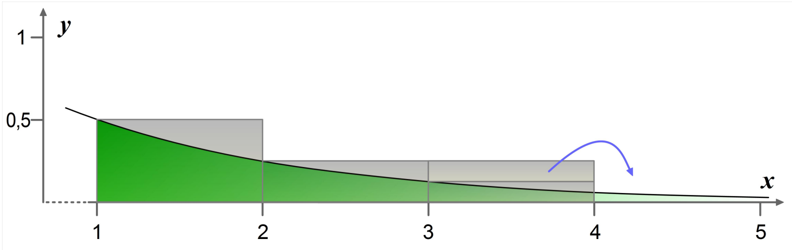

Fig. 10 The second half of the second half cut into two halves on top of each other

... and place the upper half to the right of the lower half.

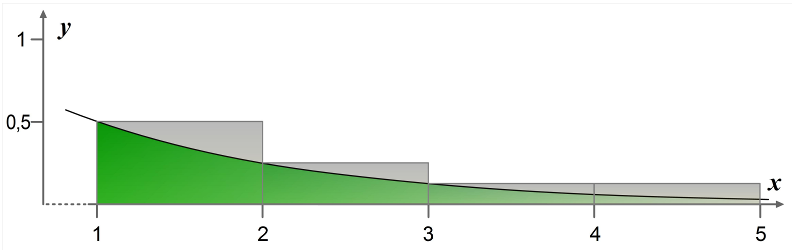

Fig. 11 The second half of the second half cut into two halves side by side

In this way, we can cover the entire infinitely wide green area. The area of the paper is larger than the green region. Therefore, the surface area of the green region must even be smaller than 1. Using integral calculus, we can calculate this:

$$\lim_{n\to\infty}\int_1^n 0.5^x\,dx = \lim_{n\to\infty}\left[-\frac{1}{\ln(2)}\times 0.5^x\right]_1^n = \frac{1}{2\times\ln(2)} \approx 0.7213$$

With this rather simple visualization, we can understand how something — like the surface area here — can keep getting larger without becoming infinitely large. That happens, for example, when what is being added becomes smaller in a suitable way.

"In a suitable way" here means not only that what is added per unit on the x-axis must go to 0. It must also go to 0 "fast enough." An example of "not fast enough" is the function $f(x) = \frac{1}{x}$. If the left integration limit is, for example, 1 and the right limit goes to infinity, then what is added per unit on the x-axis does go to 0, but the total area still goes to infinity.